Corrections

FIRST STEPS IN GEOMETRY

BY

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AND

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PREFACE

THIS book is intended to be an introduction to elementary geometry. It aims to make clear by illustrations, definitions, and exercises the exact meaning of the straight line, parallel lines, axial and central symmetry, loci of points, equal figures, equivalent figures, similar figures, and measurements of lines, surfaces, and solids.

It aims also to make the learner familiar with the most important theorems, and to teach him to draw, with instruments and free-hand, accurate figures both plane and solid.

The pupil who begins the study of any of the common textbooks in geometry with a clear knowledge of the subject matter of this book, and with skill in drawing geometric figures will have a great advantage over the pupil who begins the study without such knowledge and skill; for he can then give his whole attention to the processes of abstract reasoning employed in the demonstrations of propositions.

We acknowledge our indebtedness to Dr. Francis K. Ball of Phillips Exeter Academy, who has read all the proof sheets, and given us the benefit of his criticisms.

A box of drawing instruments can be had of the publishers at a moderate price. The student should also have a small T square and a block of paper.

G. A. WENTWORTH.
G. A. HILL.

APRIL, 1901.

CONTENTS

FIRST STEPS IN GEOMETRY

CHAPTER I

GEOMETRIC MAGNITUDES

1. Bodies. This block of wood (Fig. 1) is called a **body,** or a **solid,** because it occupies space, or has **extension.**

The block is also called a **cube,** on account of its shape or form.

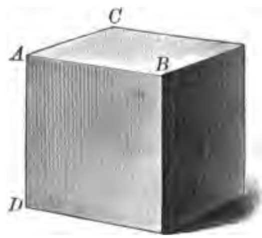


Fig. 1.—A Cube.

The cube extends in three principal directions:

from A to B (left to right),
from A to C (front to back),
from A to D (top to bottom).

The distances from A to B, from A to C, and from A to D are called the **dimensions** of the cube.

One of the dimensions is called the **length,** another the **breadth,** and the third the **thickness.** The distance from top to bottom is also called the **height,** or the **depth.**

Every body has three dimensions. If the dimensions differ in magnitude, the longest one is usually called the length.

Point out and name the dimensions of this book.

Name the dimensions of this room.

2. A cube is bounded by its **faces**; the faces are bounded by the **edges**; and the edges are bounded by the **corners.**

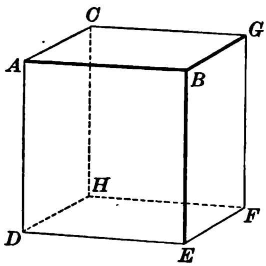


FIG. 2.—A Cube in Outline.

Fig. 2 represents a cube drawn in outline.

How many faces has the cube?

How many edges?

How many corners?

The faces are called **surfaces.**

Each face has two dimensions, and only two dimensions, length and breadth, or length and height.

Name the dimensions of the top face.

Name the dimensions of one of the side faces.

The edges of the cube are **lines.**

A line has one dimension, and only one dimension, namely, length.

Each edge is the common boundary of two faces, or the **intersection** of two faces.

The corners of the cube are **points.**

A point has position, but no extension; it has no length, breadth, or thickness.

How many edges meet or intersect at each corner?

3. A point, as defined in geometry, is not visible. It is represented to the eye on paper by a small black dot, as shown in Fig. 3.

A line is represented on paper usually by a narrow mark, or by a series of short dashes placed end to end, as shown in Fig. 3.

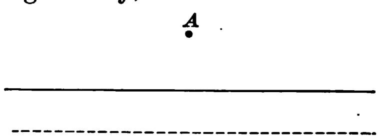


FIG. 3.

4. Straight Lines. Lines are of two kinds, **straight** and **curved.** The edges of a cube are straight lines. A string stretched tight represents a straight line. A string allowed to hang loose takes the form of a curved line, or curve.

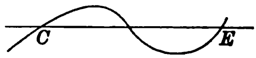


FIG. 4.

Fig. 4 represents a straight line and a curved line, both passing through two fixed points, C and E. Suppose that these lines are made to revolve about the fixed points C and E. The curved line will take different positions, but the straight line will keep the same position.

A curved line changes in position when made to revolve about two of its points.

A straight line does not change in position when made to revolve about two of its points.

Therefore, it follows that

Two points determine a straight line; that is, fix its position.

It also follows that

Only one straight line can be drawn from one point to another.

A point C (Fig. 5) in a straight line divides the line into two parts, CA and CB. All points situated on the part CA are said to have the **same direction** from the point C; and all points situated on the part CB are also said to have the same direction from the point C.

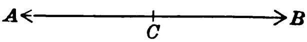


FIG. 5.

The parts CA and CB are said to have **opposite directions** from the point C.

Every straight line may be regarded as extending indefinitely in either of two opposite directions.

Another truth characteristic of a straight line in distinction from a curved line is illustrated in Fig. 6.

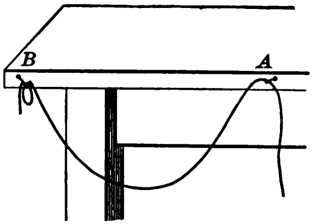


FIG. 6.

A string is fastened by one end at *B*, and the other end is passed over a nail at *A*. At first the string hangs in the form of a curve. If we pull the free end of the string, the part between *A* and *B* gets shorter and shorter; when it is as short as possible, it is a straight line. Hence,

A straight line is the shortest line from one point to another.

The way carpenters apply this truth for the purpose of drawing straight lines on wood by means of a chalk-line is shown in Fig. 7.

FIG. 7.

The **distance between two points** means the length of the straight line which joins them.

For drawing straight lines on paper a smooth piece of wood with a straight edge, called a **ruler**, is used.

1. Mark four points on paper and connect them by as many straight lines as possible.

2. Draw a straight line free-hand and test it with the ruler.

5. Plane Surfaces. Let us examine the faces of the cube more closely. What kind of surfaces are they? The answer is, **plane surfaces,** or **planes.** They belong to the same class of surfaces as the floor or the blackboard.

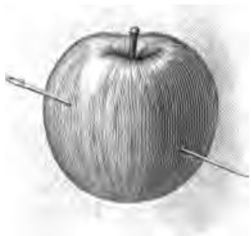


FIG. 8.

On the other hand, the surface of an apple (Fig. 8) is a **curved surface.** Many lead-pencils have curved surfaces. Probably you can find other examples of plane and curved surfaces here in the room. Now what is the difference between a plane surface and a curved surface?

Observe the blackboard. You can choose any two points on its surface and then draw on the surface a straight line from one point to the other. Can you do this on the surface of an apple? If you imagine a straight line joining two points of the surface of the apple, this line will not lie on the surface, but pass directly through the apple (Fig. 8).

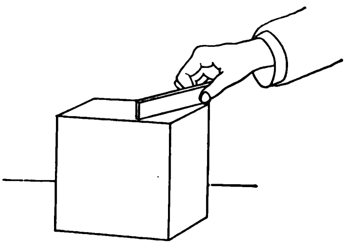


FIG. 9.

A **plane surface** is a surface such that, if a straight line is drawn between any two points in it, this line will be wholly in the surface. A surface that will not stand this test is called a **curved surface.**

To test whether a surface is plane (or *flat*, as it is commonly called), hold against it in various positions the straight edge of a ruler and see if the edge has unbroken contact with the surface (Fig. 9).

Apply this test to the face of a wooden cube.

6. Plane Figures. A plane surface, bounded by one or more lines, is called **a plane figure.**

The faces of a cube are plane figures, and each face is bounded by four straight lines.

In a true cube all these lines are equal in length. The edges of this cube look equal, but a closer test may show slight differences in length. How can a test be applied?

First Method. Hold a strip of thin paper against an edge and bend the strip where it touches the ends of the edge. Then apply the strip to the other edges; and in each case observe whether the two creases on the paper coincide with the ends of the edge.

FIG. 10.

Second Method. Apply to the edges the points of an instrument called **dividers** (Fig. 10). It has two metal legs which end in sharp points, and can turn about a pivot. The distance between the points can be regulated at pleasure by turning a screw-head, *A*. Open the legs till the points will just touch the ends of one edge of the cube; then the distance between the points is just equal to the length of this edge. Apply the points to the other edges, taking care not to change their distance apart, and thus test whether all the edges are exactly of the same length.

If you find small variations in the lengths of the edges, either you have not been careful in your work, or the cube has not been properly made and is not really a cube.

1. Draw with a ruler a straight line of any convenient length. Then with ruler and dividers draw a straight line twice as long; four times as long.

2. Draw a straight line which appears to you equal to one edge of a cube. Then test the equality by means of dividers.

7. Let us now make on paper a drawing or diagram of one face of the cube. Place the cube on the paper and trace with a pencil the outline of that face which lies on the paper (see Fig. 11).

The diagram thus made, *ABCD* (Fig. 12), is a true representation of the face of the cube both in size and in shape. It is **geometrically equal** to the face.

In fact, while constructing this diagram we have been giving an illustration of what is meant by **geometric equality.**

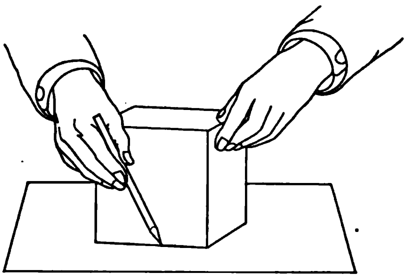


FIG. 11.

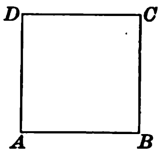


FIG. 12.

Geometric equality does not mean agreement in size only, or agreement in shape only, but agreement in both size and shape. Therefore,

Two plane figures are equal if one of them can be so placed on the other that they coincide in all their parts and form a single figure.

Now, if our cube has been accurately made, we can place the faces one after another on the diagram *ABCD* already made, so that they will exactly coincide with *ABCD*. Therefore,

1. The faces of a cube have the same size.
2. The faces of a cube have the same shape.

In other words, the faces of a cube are **equal figures.**

8. Examine one face of a cube as represented by the diagram *ABCD* (Fig. 13). We know that it is a plane figure bounded by four equal straight lines. But this description is not complete. The plane figure *EFGH* (Fig. 14) is also bounded by four equal straight lines, yet in shape it differs greatly from a face of a cube.

Take a narrow strip of cardboard to represent a straight line and bend it so as to make a square frame in shape like the figure *ABCD*. You can easily change the shape into that of the figure *EFGH*, or into other shapes.

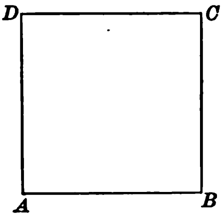


FIG. 13.

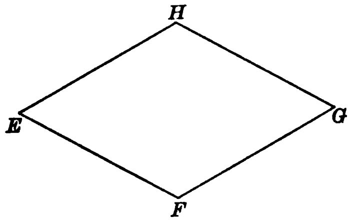


FIG. 14.

If you reflect a little, you will see that these differences in shape arise from the different ways in which two straight lines can meet to form a corner. On the cube they meet so as to form what are called square corners, or right angles. Right angles are very common. You can easily find them on a sheet of paper or an envelope, or on the floor, sides, and ceiling of a room. Very likely you know a right angle the instant you see it. But to define a right angle is not so easy a matter. We cannot hope, however, to gain a sound knowledge of geometry unless we know the exact meaning of every word which we use. We must then try to define a right angle.

Let us first explain the meaning of the word **angle**.

9. Angles. When two straight lines meet they form an **angle.** The point of meeting is called the **vertex** of the angle. The lines are called the **sides** of the angle. Thus, the straight lines AB, AC (Fig. 15), meeting at A, form the angle BAC, or the angle a. Observe carefully both ways of naming the angle.

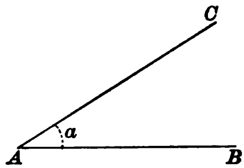


FIG. 15.

The legs of the dividers (p. 6) may be regarded as representing the sides of an angle, the pivot being the vertex. If you open the legs a little, the angle which they form is small; open the legs more and the angle becomes larger. Evidently angles differ in **magnitude.**

Equal angles are angles which can be so placed that their vertices coincide and their respective sides coincide.

This definition is in full agreement with the idea of geometric equality already explained on page 7.

When two straight lines intersect (Fig. 16), four angles are formed. Name them, using the letters in Fig. 16.

Name two angles in Fig. 16 unequal in magnitude.

Are any of the angles in Fig. 16 equal in magnitude?

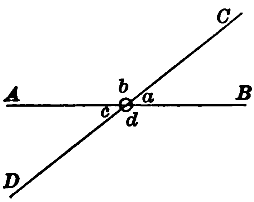


FIG. 16.

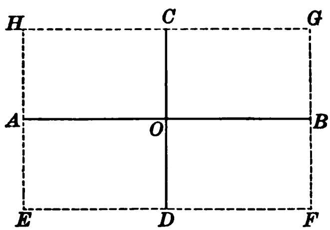


FIG. 17.

10. Right Angles. Suppose that two straight lines AB, CD (Fig. 17) intersect each other so that the four angles formed

are all equal in magnitude; then each angle is called a **right angle,** and the two lines are said to be **perpendicular** to each other.

It is very easy to make four right angles, using only your hands and a sheet of paper, *EFGH* (Fig. 17).

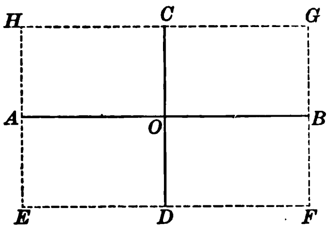


FIG. 17.

Fold over the paper so that the edge *EH* shall coincide with the edge *FG*. Unfold it and you will have made a crease or line, *CD*. Then fold the paper so that the edge *EF* shall coincide with the edge *HG*. Unfold again, and you have the crease *AB*. The two creases meet at the point *O* and form four right angles.

You know that these angles are all equal, because in the act of folding they are made to lie upon one another and coincide; and *this is the test of geometric equality.*

The edges of a cube when they meet form right angles.

The faces of a cube, therefore, are plane figures that have four right angles and are bounded by four equal straight lines.

The faces of a cube are called **squares**; and a cube may be defined as a solid bounded by six equal squares.

Squares may differ in size but not in shape.

Every square has four right angles, and is bounded by four equal straight lines, called the sides of the square.

EXERCISES

1. How many right angles are there on any one face of a cube?
2. How many right angles are there at each corner?
3. How many right angles are there on a cube?
4. How many edges are perpendicular to any one edge?
5. How would you test with pencil and paper whether the four angles of a face of a cube are all equal?

11. Parallel Lines. It will be shown in Chapter II that two straight lines perpendicular to a third straight line cannot meet however far they extend (Fig. 18).

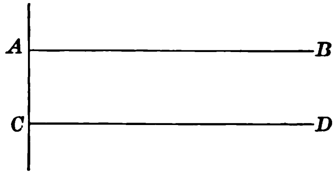


FIG. 18.

Two straight lines that lie in the same plane and cannot meet however far they extend are called parallel lines.

Two straight lines to be parallel *must lie in the same plane.*

It follows that the opposite sides of a square are parallel, for both of them are in the same plane and perpendicular to the other two sides.

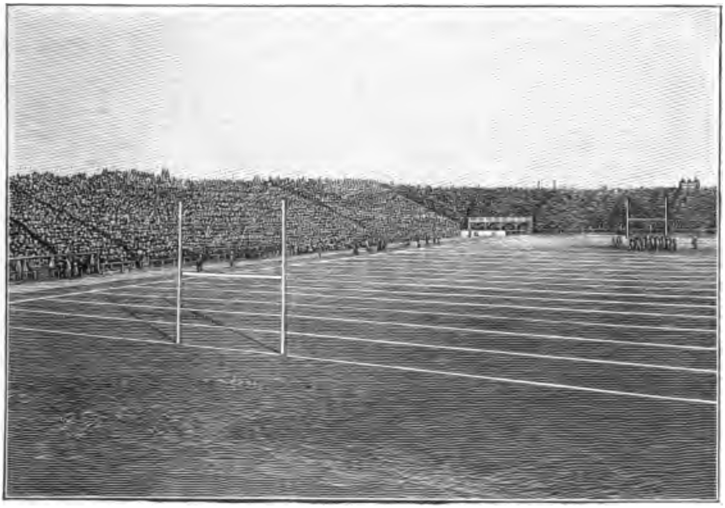


FIG. 19.—A Football Ground.

The opposite edges of a ruler, the lines on ruled paper, and the lines on a football ground are examples of parallel lines.

12. Parallel Lines and Planes. A straight line and a plane are said to be parallel if they cannot meet however far they are produced.

Two planes, also, are said to be parallel if they cannot meet however far produced.

Find examples illustrating these definitions by examining

1. The surface of a cube.
2. The surfaces that bound the room.

13. A Straight Line Perpendicular to a Plane. Lay a sheet of paper on the table and draw a straight line, AOB (Fig. 20). Hold a pencil perpendicular to AOB, with the point at O. Can the pencil be held in more than one position perpendicular to AOB at O?

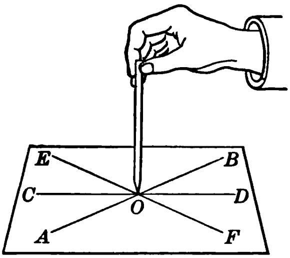


FIG. 20.

Draw another straight line, COD, and hold the pencil perpendicular to both lines AB and CD at O. Is more than one position of the pencil perpendicular to AB and CD at O possible?

The pencil, if held perpendicular to two straight lines drawn through O, *must be held in one position and in one only*. In this position the pencil is said to be perpendicular to the plane of the paper.

A straight line is perpendicular to a plane when it is perpendicular to any two straight lines that can be drawn in the plane through the point of its intersection with the plane.

Give examples of straight lines perpendicular to planes

1. On a cube. 2. In the room. 3. In a bookcase.

14. Two Planes Perpendicular to Each Other. Take a piece of cardboard and bend it so that it represents two planes intersecting along a straight line, AB (Fig. 21). From any point O in AB draw perpendicular to AB the straight line OC in one plane and the straight line OD in the other.

When the planes are placed as in Fig. 21, the angle COD is clearly *not* a right angle. But in Fig. 22 we see two

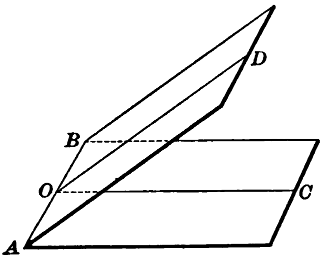


FIG. 21.

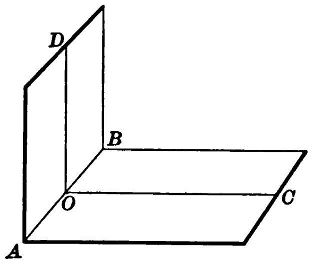


FIG. 22.

planes so placed that the angle COD is a right angle. In this position the planes are perpendicular to each other.

Two intersecting planes are perpendicular, if two straight lines, one in each plane, perpendicular to the line of intersection of the planes at the same point form a right angle.

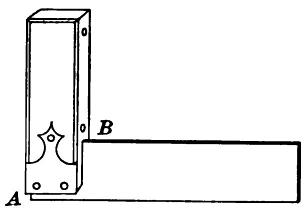


FIG. 23. — A Try Square.

A carpenter tests whether two plane surfaces are perpendicular by applying to them an instrument called a **try square** (Fig. 23).

Give examples of perpendicular planes

1. On a cube. 2. In the room. 3. In a bookcase.

15. Vertical and Horizontal Lines and Planes. A thread suspended from a fixed point (Fig. 24) and supporting a piece of lead or other metal is called a **plumb line.** A plumb line, when at rest, is said to have a **vertical direction.**

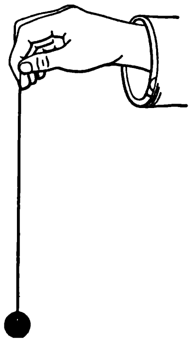


FIG. 24.

A straight line or a plane, parallel to a plumb line, is called a **vertical line or plane.**

A straight line or a plane, perpendicular to a plumb line, is said to be **horizontal.**

Straight lines and planes which are neither vertical nor horizontal are said to be **inclined.**

The surface of still water, if small in extent, is nearly horizontal; but the surface of the ocean or of a large lake is curved, because the earth is round.

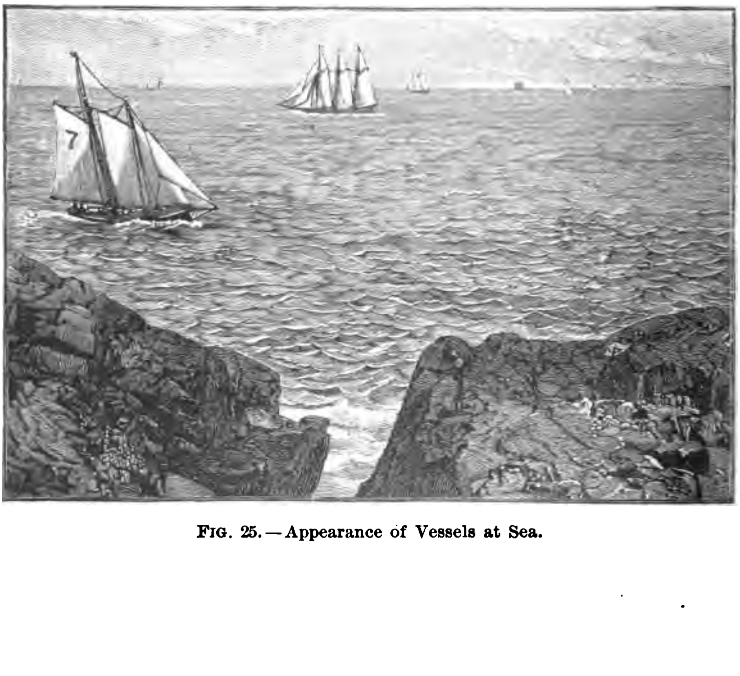


FIG. 25.—Appearance of Vessels at Sea.

EXERCISES

1. Examine the floor, walls, and ceiling of the room, and point out vertical lines and planes. Point out also horizontal lines and planes.

2. What kind of a plane is the football ground in Fig. 19, p. 11? How would you describe the five-yard lines? the goal posts?

3. How would you describe the surface of a pond when the water is at rest (Fig. 26)? What kind of line does a pole floating on the water represent? Will it make any difference if the wind blows the pole into a new position?

4. What is there in the appearance of the vessels in Fig. 25 which shows that the surface of the ocean is curved?

5. Point out vertical lines, vertical planes, horizontal lines, horizontal planes, inclined lines, and inclined planes on your desk.

6. What kind of plane surface is represented by a door? Does it make any difference whether the door be open or shut?

7. Place a cube upon the table. In this position, how many of its edges are vertical? How many are horizontal? How many of its faces are vertical? How many horizontal?

FIG. 26. — Surface of Still Water.

8. Hold a cube so that four edges shall be horizontal and all the other edges inclined to the horizon.

9. Hold a cube so that all its edges shall be inclined to the horizon.

10. Describe the positions of the hands of a clock at 8 A.M.

11. Mention the times of day when the hour hand of a clock is horizontal. What is the position of the minute hand at these times?

16. Plumb Rule. The walls of buildings should be vertical; for if this is not the case, they are in danger of falling.

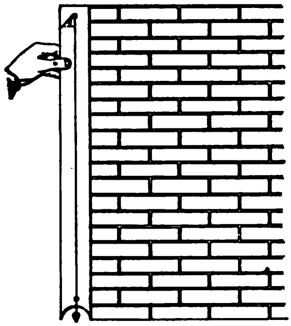


FIG. 27. — Testing a Brick Wall.

Masons, when constructing a brick wall, test whether it is vertical by holding against it a **plumb rule** (Fig. 27). The edges of the plumb rule are made parallel, and a straight line, called the test line, is drawn on the wood exactly midway between the edges. A plumb line is suspended at *A*.

The workman holds one edge of the plumb rule against the wall to be tested; if the wall is vertical, the plumb line will coincide with the test line on the wood. To complete the test, the experiment is repeated by holding the opposite edge of the rule against the wall.

17. Spirit Level. To test whether a surface is horizontal, a **spirit level** is the instrument in common use. It consists of a wooden bar having a plane surface on its lower side, and on its upper side a glass tube slightly curved and nearly full of alcohol. The space not filled with alcohol appears as a bubble of air. When the bottom of the level is horizontal, the bubble will stand exactly at the middle of the tube.

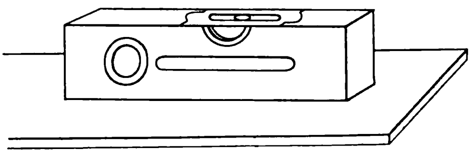


FIG. 28. — Testing a Horizontal Plane.

Take a level and test whether the surface of a table is horizontal. Is it sufficient to apply the level in only one position to the surface of the table?

18. Development of the Surface of a Cube. Let us make a diagram of the entire surface of a cube, as it would appear if spread out upon a plane surface.

Place the cube on a sheet of paper and trace the outline *ABCD* (Fig. 29) of the face which lies on the paper. Turn the cube over the edge *CD* till another face lies on the paper. Trace its outline *CDEF.* Similarly trace the outlines of two more faces, *EFGH* and *GHIK.* Then place the cube as it was when standing upon *CDEF* and trace the two remaining faces *CFLM* and *DENO* by turning the cube over the edges *CF* and *DE.*

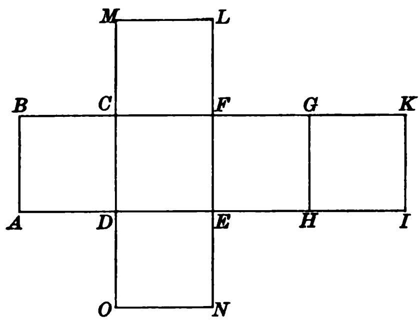


Fig. 29.—Development of the Surface of a Cube.

The entire diagram, consisting of six squares lying in one plane, is called the **development of the surface of the cube.**

If the diagram is made on thin cardboard, and if the squares are then properly bent about their sides, and their loose edges fastened together, a **model** of a cube will be the result.

Instead, however, of constructing the development of the surface as just explained, it is better to make use of a ruler, dividers, and an instrument called a **triangle.** The triangle is shown in the diagram on the next page.

19. Ruler and Triangle. A ruler and a triangle (Fig. 30) are used together for the purpose of drawing rapidly straight lines parallel or perpendicular to one another. Both instruments have straight edges, and the edges *CD* and *ED* of the triangle should be exactly perpendicular to each other.

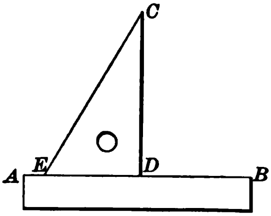


FIG. 30. — Ruler and Triangle.

Place the ruler and the triangle as seen in Fig. 30, slide the triangle along *AB*, and in different positions draw straight lines along the edge *CD* of the triangle. These lines will be perpendicular to *AB* and parallel to one another.

We proceed to explain the method of constructing a model of a cube with the aid of ruler, triangle, and dividers.

Take a piece of cardboard. Draw a straight line, *AE* (Fig. 31). Lay off with dividers the lengths *AB*, *BC*, *CD*, *DE*, each equal to an edge of the cube. Complete the figure, using ruler and triangle for drawing the lines, and the dividers for measuring lengths. Draw small laps on seven edges, as seen in Fig. 31. The laps are used to fasten together the edges after folding.

Cut out the figure, and fold on the lines that are represented as dotted lines in the figure. Apply mucilage or glue sparingly to the laps, and fold so that the laps shall come on the inside of the faces.

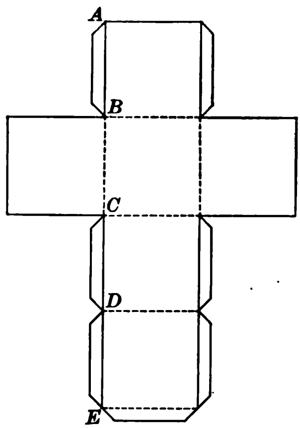


FIG. 31.

EXERCISES

1. How many cubes, all of the same size as the model, must be put together to make a cube of larger size than the model?

2. How do the lengths of the edges of the model and new cube compare?

20. Units of Length. You are now prepared to understand the great convenience of employing **units of length.**

The standard units of length in common use are:

the **inch** (in.), the **foot** (ft.), the **yard** (yd.), and the **mile.**

12 in. = 1 ft.; 3 ft. = 1 yd.; 5280 ft. = 1 mile.

The inch is divided into halves, quarters, eighths, and sixteenths.

A line is **measured** by finding the number of units of length it contains. The number of units with the name of the unit is called the **length of the line.**

Short lines are measured by a graduated ruler.

A part of a graduated ruler is shown in Fig. 32.

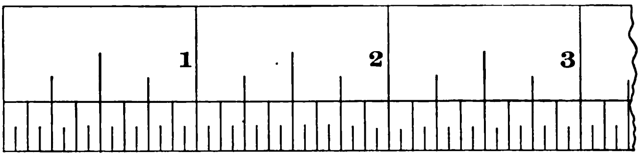


FIG. 32.—Ruler Graduated to Sixteenths of an Inch.

As exercises merely in estimating lengths by the eye, give the length of your teacher's desk; the length of the room; the width of the room; the length of the blackboard.

Pacing is a rough but quick way of measuring distances. The two following exercises should be done by you before the next recitation, and the results reported at that time.

EXERCISES

1. Find the average length of your pace. To do this take 10 steps as you naturally walk, measure the distance in feet, and divide this distance by 10. Repeat, and if the results differ, find the mean result by adding the two results and taking half the sum.

2. Find, by pacing, the distance from your home to your school.

Some exercises in measuring lines will now be given. In each case, begin by trying to estimate the length to be measured as well as you can, then measure the line correct to a

LINE TO BE MEASURED.	ESTIMATED LENGTH.	MEASURED LENGTH.	DIFFERENCE.
Edge of Cube . .			
Length of Page .			
Etc.			

sixteenth of an inch with a graduated ruler, and make a record of your work in the form of a table, as shown above.

EXERCISES

1. Measure the edge of the cube given you.
2. Measure the length of this page.
3. Measure the width of your desk.

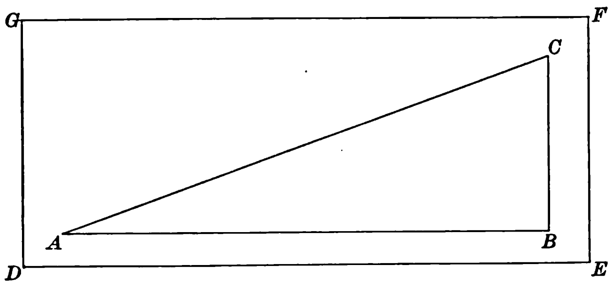


FIG. 33.

4. Measure the length of a printed line on the page.
5. Measure the line AB in Fig. 33.
6. Measure the line BC in Fig. 33.
7. Measure the line AC in Fig. 33.
8. Measure the line DE in Fig. 33.
9. Measure the line EF in Fig. 33.

21. The Metric System of Units. The metric units of length are the **millimeter** (mm), the **centimeter** (cm), the **decimeter** (dm), the **meter** (m), and the **kilometer** (km).

They form a part of the *Metric System* of units, which is throughout a decimal system.

$$10 \text{ mm} = 1 \text{ cm}, \quad 10 \text{ dm} = 1 \text{ m},$$
$$10 \text{ cm} = 1 \text{ dm}, \quad 1000 \text{ m} = 1 \text{ km}.$$

A ruler graduated to millimeters is shown in Fig. 34.

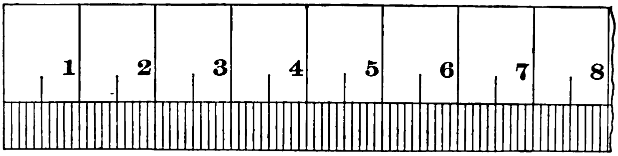


FIG. 34.—Ruler graduated to Millimeters.

A length expressed in any metric unit is reduced to the next *smaller* unit by *multiplying* the number by 10, and to the next *larger* unit by *dividing* the number by 10; the first reduction is made by moving the decimal point one place to the right, the second by moving it one place to the left.

Thus $\quad 3.25 \text{ m} = 32.5 \text{ dm} = 325 \text{ cm} = 3250 \text{ mm}.$

And $\quad 6452 \text{ mm} = 645.2 \text{ cm} = 64.52 \text{ dm} = 6.452 \text{ m}.$

In recording a measurement in metric units, use *only one* unit, and employ the decimal point when necessary. For example, if the length of an edge of a cube is 9 centimeters and 6 millimeters, record it either as 9.6 cm, or 96 mm.

EXERCISES

1. How many millimeters are there in 1 dm? in 4 dm? in 1 m?
2. How many centimeters are there in 1 m? in 6 m? in 35 mm?
3. How many decimeters are there in 7 m? in 20 cm? in 200 mm?
4. How many meters are there in 50 dm? in 225 cm? in 800 mm?

Perform again the first seven exercises on page 20, using metric units.

22. The Square Prism. The solid in Fig. 35 is called a **square prism.**

How many faces has it? How many edges has it? How many corners has it?

The upper and lower faces are called the **bases.**

The other faces are called the **lateral faces.**

The edges connecting the bases are called the **lateral edges.**

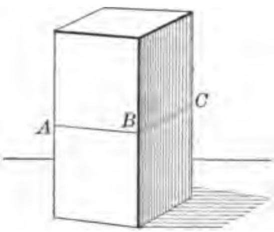


FIG. 35.—A Square Prism.

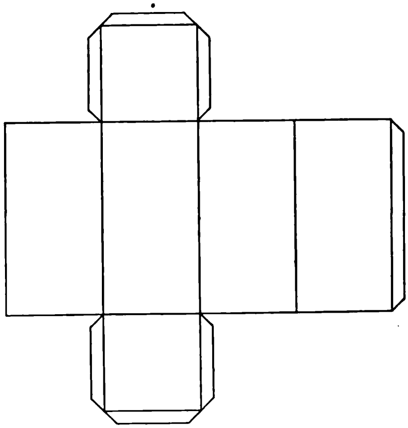


FIG. 36.—Development of a Square Prism.

The lateral edges all have the same length. The edges of the bases have the same length. But a lateral edge and a base edge do not have the same length, hence the lateral faces are not squares.

The faces have four right angles, but they do not have four equal sides. They are called **rectangles.**

Make a model of a square prism precisely as you made a model of a cube (p. 18). Make each lateral edge 7 cm long and each base edge 4 cm long.

If you cut a square prism into two equal parts by a plane surface parallel to the bases, what kind of solids will the parts be?

23. The Rectangular Prism. The solid in Fig. 37 is called a **rectangular prism.** It is also called a **rectangular parallelopipedon.**

In what respects is it like a square prism? In what respects does it differ from a square prism?

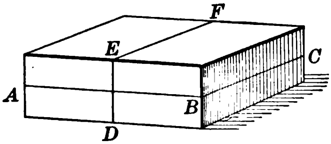


FIG. 37. — A Rectangular Prism.

Are all the angles on the prism right angles?

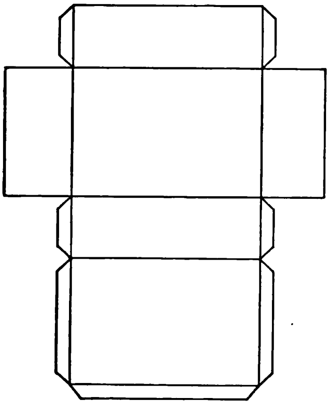


FIG. 38.

Are all the faces of the prism rectangles?

Make a model of a rectangular prism.

Make the length 12 cm ($4\frac{1}{2}$ in.), the breadth 8 cm (3 in.), and the height 4 cm ($1\frac{1}{2}$ in.).

The development of the surface is shown in Fig. 38.

A prism, either square or rectangular, is a very common form in which things are made. A shelf of a bookcase is a rectangular prism. A sheet of paper, when its thickness is considered, is a rectangular prism.

Give other examples of prisms.

EXERCISES

1. Point out on your model three pairs of parallel faces.
2. Point out on your model three groups of parallel edges.
3. If the prism is cut in two by a plane through AB and BC which are parallel to the base, and also by a plane through DE perpendicular to the plane ABC, and EF perpendicular to ED, what kind of solids are the parts?

24. Triangular Prisms. If a square prism (Fig. 39) is cut in two along the lateral edges AB and CD, it will be divided into two parts, equal in size, called **triangular prisms.**

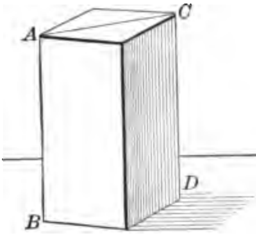


FIG. 39. — Square Prism.

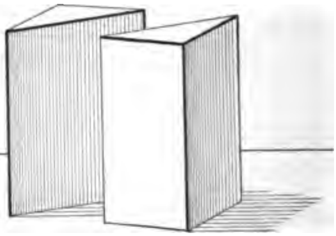


FIG. 40. — Triangular Prisms.

Triangular prisms are represented in Fig. 40.

The bases of a triangular prism are plane figures bounded by three straight lines.

Plane figures which are bounded by three straight lines are called **triangles.**

A triangle having a right angle is called a **right triangle.**

EXERCISES

1. Point out on the square prism parallel planes; perpendicular planes; parallel lines; perpendicular lines; lines parallel to planes; lines perpendicular to planes.
2. Hold the prism so that one of the lateral faces shall be horizontal. Then describe the position of all the other faces, and also of all the edges.
3. Hold the prism so that all of its edges, and also all of its faces, shall be inclined to the horizon.
4. Are the bases of a triangular prism equal or unequal figures?
5. Describe how you would test their equality.
6. What kind of triangles are they?
7. How many lateral faces has a triangular prism?
8. What kind of figures are the lateral faces?
9. Compare the lateral faces in respect to magnitude.

To make a model of a triangular prism, construct the development of the surface as follows:

Draw a straight line, and on this line take

$AB = 4$ cm ($1\frac{1}{2}$ in.),
$BC = 8$ cm (3 in.),
$CD = 4$ cm ($1\frac{1}{2}$ in.).

Draw through B and C lines perpendicular to AD.

Take BE, BF, CG, CH, each 4 cm ($1\frac{1}{2}$ in.).

Draw the straight lines AF, DH, EG.

Extend BF and CH.

Take FK equal to FA, and HL equal to HD.

Draw KL.

Draw laps as shown in the figure.

Cut out the figure and fold on the required lines.

Fasten by means of the laps.

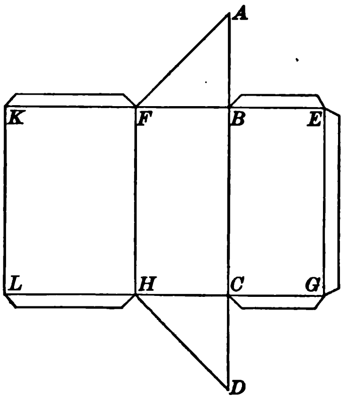


FIG. 41.

Prisms may differ in the number of edges that bound their bases; but all prisms agree in having for bases two parallel and equal plane figures, and in having for lateral faces plane figures bounded by two pairs of parallel lines.

EXERCISES

1. Point out parallel faces on your model; parallel edges.
2. Point out perpendicular faces; perpendicular edges.
3. How many right angles in all are formed by the edges?
4. How many angles are there that are not right angles?
5. Hold the model so that six edges may be horizontal.
6. Hold the model so that five edges may be horizontal.
7. Hold the model so that three edges may be horizontal.
8. Hold the model so that two edges may be horizontal.

25. Geometric Bodies. In the study of geometry a sharp distinction is drawn between the **form** of a body and the **matter** of which it is composed. The form may be that of a cube, a prism, a sphere, etc.; the matter may be wood, or iron, or glass, or some other substance.

More is implied in this distinction than appears at first sight. Take two cubes, one made of wood, the other of iron. Not only do they differ in their properties because they are composed of different materials, but it is also true that neither of them, however carefully made, is a perfect cube. Human skill is incapable of making their faces perfect planes, their edges perfect straight lines, their corners perfect points, or their angles perfect right angles.

We can, however, easily **imagine** such a thing as a perfect cube; we have only to form a mental picture of a portion of space bounded by six perfect squares, absolutely equal one to another.

This **geometric cube,** as we may call it, exists indeed only in the mind; but it is a far simpler object of study than any material cube; for it has no other properties than those which are connected with its form. And these properties when discovered can be stated as absolute truths because the cube which we study is a perfect cube.

Geometry is a science in which we study bodies with respect to form and position only. In order to make the study possible, we substitute in thought for material bodies ideal forms enclosing space, known as **geometric bodies.** When in this way a science of geometry has been constructed, the truths discovered can be applied to useful purposes by substituting for actual material bodies the ideal forms that the actual bodies most closely resemble.

REVIEW EXERCISES

1. Write the best definitions you can of the terms below. Try to make your definitions perfectly clear and to use as few words as possible.

Body	Angle	Perpendicular Lines
Surface	Right Angle	Parallel Planes
Line	Square	Perpendicular Planes
Point	Rectangle	Horizontal Line
Straight Line	Triangle	Vertical Line
Plane Surface	Right Triangle	Horizontal Plane
Plane Figure	Parallel Lines	Vertical Plane

2. When is a straight line parallel to a plane?

3. When is a straight line perpendicular to a plane?

4. Can two horizontal planes intersect each other?

5. Can two vertical planes intersect each other?

6. Can a horizontal plane intersect a vertical plane?

7. What kind of a line is the intersection of two vertical planes?

8. How many vertical lines can be drawn through a point? How many horizontal lines? How many inclined lines?

9. How many horizontal lines can be drawn in a vertical plane? How many in a horizontal plane?

10. How many vertical lines can be drawn in a horizontal plane? How many in a vertical plane?

11. Draw three straight lines so that they intersect one another in three points; in two points; in one point.

12. Through a given point how many straight lines parallel to a given straight line can be drawn?

13. Draw a straight line. Then place three points so that you can draw through each one of them a line parallel to the straight line already drawn.

14. How many pairs of parallel lines can be drawn through two given points?

15. Through a given point how many straight lines perpendicular to a given straight line can be drawn?

16. Draw a straight line, and then place three points so that only one straight line can be drawn perpendicular to the line already drawn and passing through all three points.

17. How many straight lines can be drawn from a given point to a given straight line? Describe the position of the shortest of these lines.

18. How many right angles are formed when two perpendicular lines intersect each other?

19. Mention the times of day when the hands of a clock form a right angle.

20. Is the angle formed by the hands of a clock greater or less than a right angle at 10 A.M.? at 4 P.M.?

21. How many times during 24 hours do the hands of a clock make the same angle as at 10 A.M.?

22. What kind of a straight line is a line running north and south? a line running east and west?

23. Draw a straight line, AB, 4 in. long. At A erect a perpendicular, $AC = 1\frac{1}{2}$ in. At B erect a perpendicular, $BD = 4\frac{1}{2}$ in. Join CD; measure its length; write the value of the length by its side.

Work this exercise first free-hand, and then with instruments.

24. Draw a straight line, $AB = 5$ in. Erect at B a perpendicular, $BC = 3$ in. Draw through C a straight line, $CD = 1$ in., and parallel to AB. Join AD and measure its length.

Work this exercise first free-hand, and then with instruments.

CHAPTER II

GEOMETRIC MAGNITUDES AND MOTION

26. Path of a Moving Point. When we move the point of a pencil along a sheet of paper the point of the pencil leaves a black trace which we call a line. It is not a line, however, but a long, narrow body.

Imagine the point of the pencil to be a point as understood in geometry; then the trace will represent a *geometric line*, having length as its sole dimension.

We may regard any line as made or generated by a moving point, and define a line as *the path of a moving point*. This way of regarding a line is often very useful.

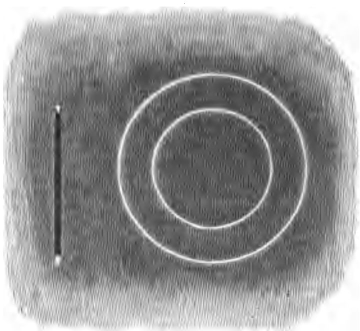


FIG. 42.

The generation of a line by a moving point is forcibly illustrated when a small luminous body, as, for example, the red-hot end of a poker, is put in rapid motion in a dark room. The end of the poker appears to be changed to a line of fire. The following experiment is still more striking.

Mount a Chinese incense stick on an axis (Fig. 42), so that its ends are unequally distant from the axis. Light the ends and set the stick in rapid rotation. Instantly in place of the luminous points two bright rings are seen.

This effect is explained by the fact that the impression of light produced on the retina of the eye lasts long enough for the luminous point to make one revolution, and thus renew the impression before it has time to die out.

27. Path of a Moving Line. A moving line, in general, generates a surface.

Mount a glass tube upon an axis so that the tube is perpendicular to the axis, and the axis passes through the middle point of the tube (Fig. 43). When at rest, the tube represents a straight line. But when set in rapid rotation, the tube will appear to be transformed into a well-known plane figure called a **circle.** This plane figure is the path traversed by the tube during each revolution about the axis. Reduce in thought the tube to a geometric line, and its path will be reduced to a geometric magnitude of two dimensions, that is, to a surface.

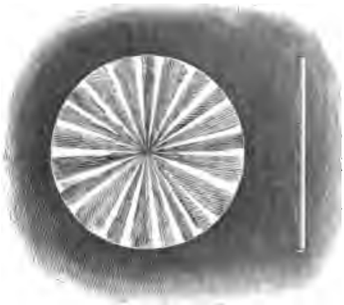


FIG. 43.

28. Path of a Moving Surface. A moving surface, in general, generates a solid. If the surface *ABCD* (Fig. 44) is moved to the right to the position *EFGH*, the surface *ABCD* will generate the solid *AG*. The lines *AB*, *BC*, *CD*, and *DA* will generate the surfaces *AF*, *BG*, *CH*, and *DE*, and the points *A*, *B*, *C*, and *D* will generate the lines *AE*, *BF*, *CG*, and *DH*.

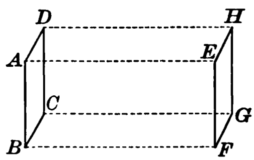


FIG. 44.

29. The Circle. Suppose that a straight line OA (Fig. 45) revolves about the point O, keeping always in the same plane, till it returns to its first position. It will generate, or describe, a plane figure, called a **circle**. The point A will describe a curved line, which is the boundary of the circle, and is called the **circumference** of the circle.

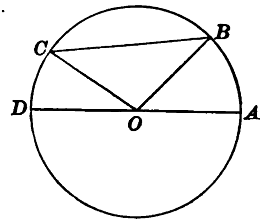


FIG. 45.

The point O about which the line revolves is called the **centre** of the circle.

The circumference of a circle is the path of a point moving in a plane at a given distance from the centre.

A straight line drawn from the centre to the circumference is called a **radius** (plural, **radii**).

A straight line drawn through the centre and extending to the circumference in both directions is called a **diameter**.

A straight line joining any two points of a circumference is called a **chord**.

A part of a circumference is called an **arc**.

Every diameter divides the circle into two equal parts, called **semicircles**. Every diameter divides the circumference into two equal parts, called **semicircumferences**.

EXERCISES

1. Name in Fig. 45 a radius, a chord, an arc, a diameter.
2. What is true of all radii of the same circle?
3. What is true of all diameters of the same circle?
4. Compare a radius with a diameter as regards length.
5. Compare a diameter and a chord as regards length.
6. In Fig. 45, what is described by any point in the line OA?
7. What is represented by the tire of a carriage wheel? by the spokes? by the part of the tire between two spokes?
8. Can you name any objects that are circular in shape?

30. Compasses. Circles are described on paper by means of **compasses** (Fig. 46). The dividers (Fig. 10, p. 6) can be changed to compasses by substituting for one of the metal points a pen or a pencil.

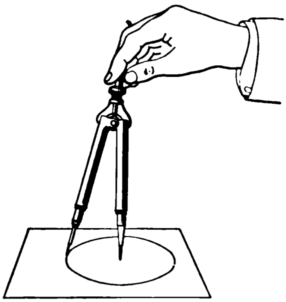


FIG. 46.

The compasses should be held between the thumb and the forefinger, as shown in Fig. 46, and the metal point should be pressed against the paper just hard enough to keep it in place.

On the blackboard, circles may be described with the aid of a string. Make a loop at one end of the string and slip it over the end of a crayon. Press the string against the blackboard at the point chosen for the centre. Keep the string stretched while you move the crayon round the centre.

FIG. 47. — How Circles are made on the Ground.

In gardens, parks, etc., it is sometimes desirable to mark out large circles. The method by which circles are commonly drawn in gardens is illustrated in Fig. 47.

31. The Cylinder, the Cone, and the Sphere. A straight line perpendicular to the axis about which it revolves generates a plane surface. In any other position the line generates a curved surface.

When a rectangle $ABCD$ (Fig. 48) revolves about one side AD as an axis, the opposite side BC generates the curved surface of a solid called a **cylinder.**

When a right triangle ABC (Fig. 49) revolves about one of the sides that form the right angle, the side opposite the right angle generates the curved surface of a solid called a **cone.**

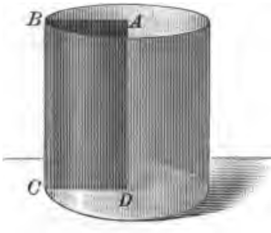


FIG. 48.

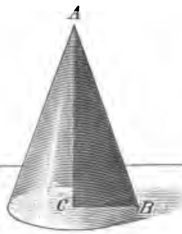


FIG. 49.

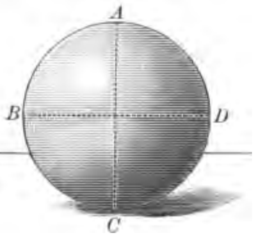


FIG. 50.

When a semicircumference ABC (Fig. 50) revolves about the diameter AC as an axis, it generates the curved surface of a solid called a **sphere.**

EXERCISES

1. What is generated by the sides AB and DC of the rectangle in Fig. 48? What is generated by the points B and C?

2. What is generated by the side BC of the triangle in Fig. 49? What is generated by a point of AB between A and B?

3. What is generated by the point B on the curve ABC in Fig. 50, as the curve revolves about the axis AC?

4. What solid is generated when a rectangular piece of cardboard is moved in a straight line perpendicular to its own plane?

5. How can a plane surface be moved so as not to generate a solid?

32. Drawing Exercises. These exercises require the use of the instruments hitherto mentioned, and also a rubber eraser to remove *auxiliary* lines, or lines which aid in constructing the figure but do not belong to the figure when completed.

If for any reason you wish to leave auxiliary lines in the figure, they should be drawn with short dashes.

EXERCISES

1. Describe a circle with a radius of 1 in.

2. Describe a circle with a radius of 3 cm.

3. Describe a circle and then draw a chord equal to the radius. Also draw a chord equal to twice the radius.

4. Describe three concentric circles, that is, circles having the same point as centre.

5. Describe two circles such that the circumference of each shall pass through the centre of the other. How do they compare in size?

6. Describe three unequal circles so that the centre of each shall lie on the circumference of one of the other two.

7. Describe four equal circles such that the circumference of each circle shall touch but not cut the circumferences of two of the other circles.

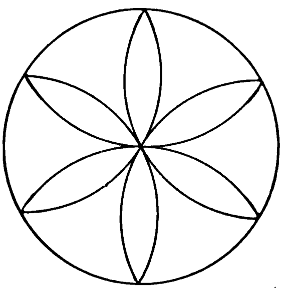


FIG. 51.

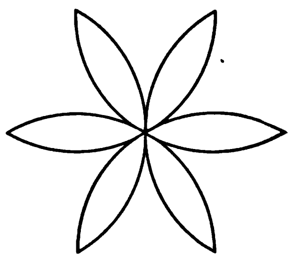


FIG. 52.

8. Construct Fig. 51. The circle and the arcs have the same radius. Then make a six-rayed leaf (Fig. 52) by erasing the circumference.

9. Construct an *equilateral triangle*, or triangle having three equal sides (Fig. 53).

Draw a straight line, AB. With A and B as centres, and the length AB as radius, describe arcs intersecting at C. Join by straight lines A to C, and B to C. ABC is an equilateral triangle.

Erase the arcs when C has been found.

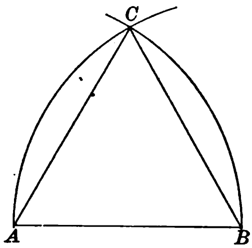


FIG. 53.

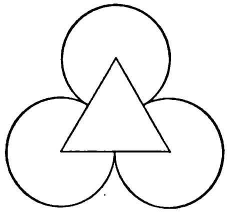


FIG. 54.

10. Construct a *trefoil* (Fig. 54).

Make an equilateral triangle. Find by trial the middle point of one side. Describe three arcs of circles with the corners of the triangle as centres, and a radius equal to half the side of the triangle.

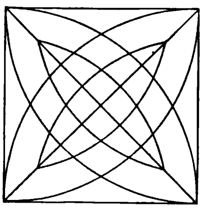


FIG. 55.

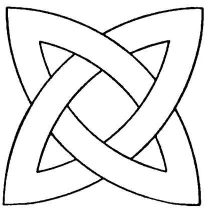


FIG. 56.

11. Construct the ornamental design shown in Fig. 56.

First make a square as in Fig. 55, and join the opposite corners.

From the four corners as centres draw arcs with the length of one side of the square as a radius; and also from the four corners as centres draw the four other arcs with a smaller radius.

Then erase lines so as to show Fig. 56.

Construct the ornamental figures given below:

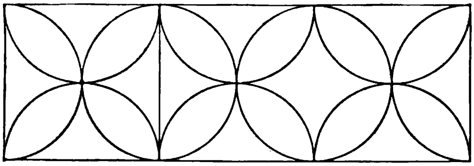


FIG. 57.

12. Construct equal adjacent squares (Fig. 57), and draw semicircles on the sides of the squares as diameters. Then erase lines so as to show the figure which appears on the right.

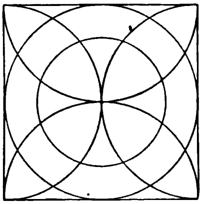


FIG. 58.

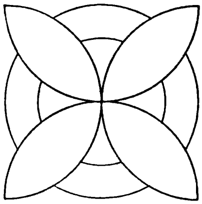


FIG. 59.

13. Construct a square, draw two concentric circles with the radius of the smaller circle two thirds that of the larger, and draw the semicircles, as shown in Fig. 58. Then erase lines so as to show Fig. 59.

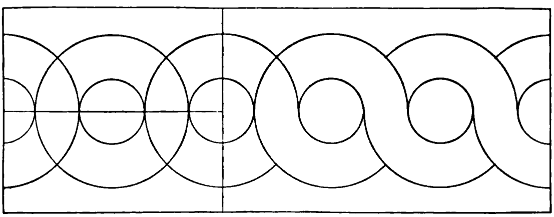


FIG. 60.

14. Draw the indicated circles (Fig. 60), with the radius of the larger ones twice that of the smaller. Then erase lines so as to show the figure as it appears on the right.

33. Loci. When a point describes the circumference of a circle it has to obey, so to speak, the law that *it must always be at the same distance from the centre of the circle.*

In general, the path of a point moving in obedience to a prescribed law is called the **locus** of the point. The plural of locus is **loci.**

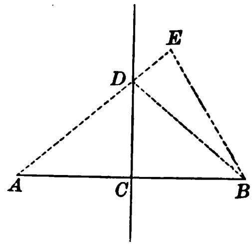


FIG. 61.

Let us consider some more examples of loci.

Suppose that a point starts at C (Fig. 61), halfway between two fixed points A and B, and describes a straight line CD perpendicular to the line AB. *The moving point will always be equally distant from A and B.* For, let us take any point D in the perpendicular, and let us imagine that the figure ACD is folded about CD through half a revolution. The line CA will fall upon the line CB, because the right angles at C are equal; the point A will fall upon the point B, because $CA = CB$; and, therefore, the line DA will coincide with the line DB, for only one straight line can be drawn between the two points D and B. Therefore, $DA = DB$.

On the other hand, any point E not in the perpendicular CD is unequally distant from A and B; for $EA = ED + DA = ED + DB$ (since DA and DB are equal), and is, therefore, greater than EB; for a straight line is the shortest line between two points. Hence,

The locus of a point so moving in a plane that it is always equidistant from two fixed points in that plane is the perpendicular erected at the middle of the straight line which joins the two fixed points.

The perpendicular DC divides the straight line AB into two equal parts, and is called the **perpendicular bisector** of AB.

34. The Ellipse. Fasten a sheet of paper to a smooth board by two pins at F and F' (Fig. 62). Put over the pins a string in the form of a loop, taking care that the loop is somewhat longer than the distance FF'. Then move the point P of a pencil, keeping it pressed against the string, as shown in Fig. 62.

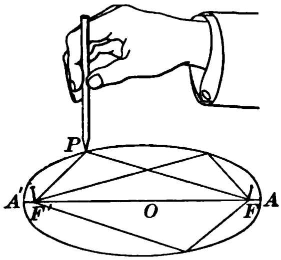


FIG. 62.

The point P will trace a smooth closed curve called an **ellipse.** The points F and F' are called the **foci** of the ellipse.

The distance $PF + PF'$ remains the same whether P is in motion or at rest.

An ellipse is the locus of a point so moving that the sum of its distances from two fixed points (the foci) is constant.

The length AA' is equal to $PF + PF'$ and is called the **major axis** of the ellipse.

The major axis bisects the curve and the area enclosed by the curve.

The less the distance between the foci the more nearly the form of an ellipse approaches that of a circumference of a circle. If the foci coincide, in other words, if only one pin is used, the curve will be a circumference.

If you stand in front of a luminous point, revolving about an axis, it will appear to describe a circumference. As you move sideways the curve will appear to change to an ellipse more and more oval in shape, and finally appear as a straight line.

The ellipse is a curve often used in cabinet work (mirrors, tables, etc.), in the arches of bridges, and in architecture. The earth and all the other planets move round the sun in elliptical curves, the sun being at one of the foci.

35. The Parabola. Let a point F (Fig. 63) and a straight line CE be fixed in position. Draw FD perpendicular to CE. Then FD is the shortest line that can be drawn from F to CE, and is called the *distance* from F to CE.

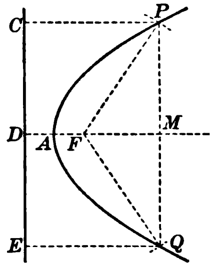


FIG. 63.

Now, suppose a point starts at A, halfway between F and D, and so moves that it is always equidistant from F and CE.

The point cannot move from A in a straight line; for there can be no straight line every point of which is equidistant from the fixed point F and the fixed straight line CE.

The point cannot describe a circumference about F as centre; for in that case its distance from CE would vary, while its distance from F would not change.

The actual path of the point is a curve PAQ, which is called a **parabola.**

A parabola is the locus of a point so moving that it is always equidistant from a given point and a given straight line.

The line AF extended is called the **axis** of the parabola, and the point F is called the **focus** of the parabola. The parabola evidently extends towards the right, both above and below the axis, as far as the point is supposed to move.

The portion of the parabola above the axis is exactly similar to the portion below the axis.

The parabola is a very graceful curve. The path of a ball, when the ball is thrown in any other direction than a vertical one, is very nearly a parabola. The same is true of a jet of water issuing from a pipe in a horizontal or inclined direction.

If a parabola is revolved about its axis, it generates a surface called a **parabolic surface.**

A parabolic surface made of metal polished on the concave side is called a parabolic reflector. When a light is placed at the focus, all the rays of light that fall on the reflector are reflected in straight lines parallel to the axis (Fig. 64).

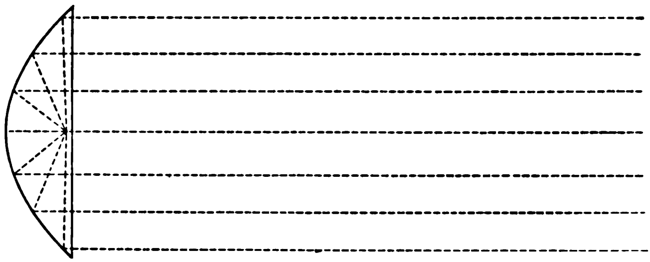


FIG. 64.

Advantage is taken of this property in the construction of the headlights of locomotives for the purpose of concentrating the light and throwing it as far as possible along the track.

FIG. 65. — Brooklyn Suspension Bridge.

The steel cables that support a suspension bridge are parabolic curves (Fig. 65).

36. The Cycloid. Consider the motion of a point P (Fig. 66) on the rim of a wheel, as the wheel rolls along a straight track.

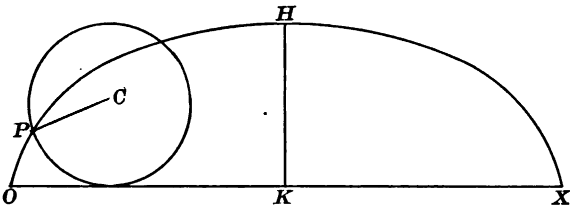


FIG. 66. — The Cycloid.

If the wheel simply revolved about its centre C, the point P would describe a circumference; if the wheel did not revolve, but was made to slide along the track, the point P would describe a straight line parallel to OX. But the wheel has both motions; it revolves about C and also moves along the track. The consequence is that the point P describes a curve $OPHX$, which is called a **cycloid.**

A cycloid is the locus of a point moving on the circumference of a circle as the circle rolls along a straight line.

The length of the cycloid from O to X is exactly four times the diameter of the generating circle.

The area of the surface between the curve OHX and the straight line OX is exactly three times the area of the generating circle.

37. The number of different curved lines that may be generated by points moving according to different laws is unlimited. But in elementary geometry the straight line and the circumference of the circle are the only lines selected for study.

38. Generation of Angles. An angle has already been defined (p. 9). Let us now regard an angle as a magnitude generated by motion. When a straight line OA (Fig. 67) describes a circle by revolving about the point O, it also generates an angle of constantly increasing magnitude. One side of the angle is OA in its first position; the other side is OB, OC, etc. If OD is perpendicular to OA, the angle generated AOD is a **right angle.** When the revolving line reaches the position OF, drawn opposite to OA, the angle generated is equal to two right angles, and is called a **straight angle.** When the revolving line has arrived at the position OG, the angle generated AOG is equal to three right angles. When the revolving line arrives at its original position OA, it has made one revolution and generated an angle equal to four right angles, or two straight angles.

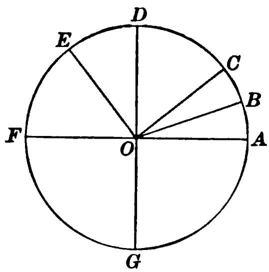


FIG. 67.

An angle less than a right angle is called an **acute angle.**

An angle greater than a right angle and less than a straight angle is called an **obtuse angle.**

The angle formed by any two straight lines may be regarded as generated by one of the lines turning about the vertex till it coincides with the other line. Hence, the magnitude of an angle depends only on the *amount of rotation* required to generate the angle.

The common units for measuring angles are:

the **degree** (°), the **minute** (′), the **second** (″).

A right angle is equal to 90°, a degree is equal to 60′, and a minute is equal to 60″.

Two angles are called **supplementary** angles, if their sum is 180°; and each is called the **supplement** of the other.

Two angles are called **complementary** angles, if their sum is 90°; and each is called the **complement** of the other.

EXERCISES

1. How many angles are formed by the two intersecting straight lines in Fig. 68? How many acute angles? How many obtuse angles? Are the acute angles equal? Are the obtuse angles equal?

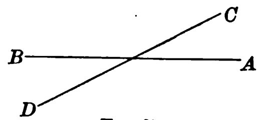


FIG. 68.

2. What kind of an angle do the hands of a clock make at 2 o'clock? 3 o'clock? 4 o'clock? 6 o'clock?

3. Mention the hours when the hands of a clock form a right angle, an acute angle, an obtuse angle.

4. In what period of time is a right angle described by the minute hand? by the hour hand?

5. How many degrees does the hour hand describe in 1 hour? in 3 hours? in 5 hours? in 24 hours?

6. By what angle must a man walking north change the direction of his motion in order to walk towards the east (Fig. 69)?

In order to walk towards the south?
In order to walk N.E.?
In order to walk S.E.?
In order to walk S.W.?

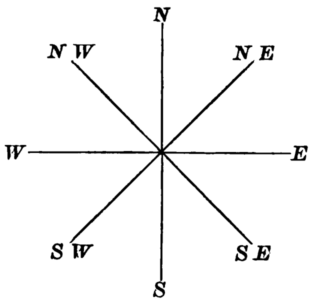


FIG. 69.

7. How many degrees are there in four right angles?

8. How many minutes are there in a right angle?

9. Reduce 10° 6′ 20″ to seconds, and 18,000″ to degrees.

10. Find the supplements of 30°, 45°, 60°, 80°, 120°, 150°.

11. Find the complements of 10°, 20°, 30°, 40°, 60°, 75°.

12. What angle has a supplement equal to itself?

13. What angle has a complement equal to itself?

14. What angle has a complement equal to half of itself?

39. Measurement of Angles. A practical method for the measurement of angles is obtained by taking advantage of a simple relation between an angle and the arc between its sides, described by the revolution of one side of the angle about the vertex till it coincides with the other side. The angle and the arc *increase at the same rate.* If the angle is doubled, the arc is doubled; if the angle is trebled, the arc is trebled, and so on.

Suppose (Fig. 70) that the angles AOB, BOC are equal. Then the arcs AB and BC must also be equal. For, if we fold AOB about OB as an axis through half a revolution, OA will fall upon OC, because the angles BOA and BOC are equal; and A will fall on C, because OA and OC are equal radii; therefore, the arc BA will coincide with the arc BC, and be equal to it in length. Hence,

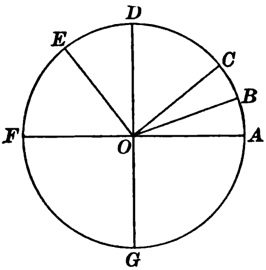


FIG. 70.

Equal angles at the centre of a circle intercept equal arcs on the circumference.

This truth is applied to the measurement of angles by dividing arcs into degrees, minutes, and seconds, precisely as angles are divided. An entire circumference, therefore, is divided into 360 equal parts called degrees. Then, to find the number of degrees, minutes, and seconds in an angle, we have only to find the number of degrees, minutes, and seconds in an arc described from the vertex of the angle as a centre, and contained between its sides. Hence, we say that *an angle at the centre of a circle is measured by its intercepted arc*, meaning that the angle contains as many angle degrees, minutes, and seconds as its intercepted arc contains arc degrees, minutes, and seconds.

40. The Protractor. On paper, angles are measured and constructed with the aid of an instrument called a **protractor.**

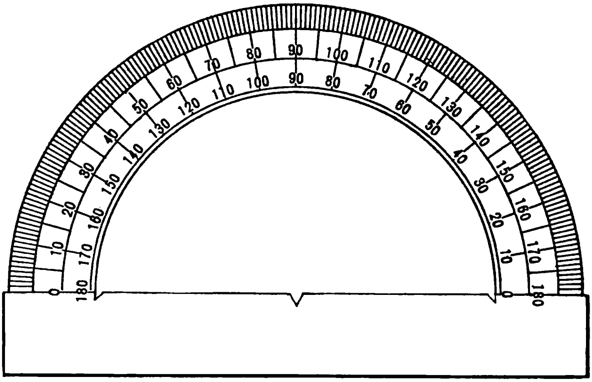


FIG. 71. — A Protractor.

A circular protractor (Fig. 71) consists of a semicircular piece of cardboard, horn, or metal. The arc is divided into degrees. In protractors made of horn or metal an open space is left round the centre to make easier the construction or measurement of angles with short sides.

To *measure* an angle, place the centre of the protractor on the vertex of the angle, and make the diameter of the protractor coincide with one side of the angle; then read on the divided edge of the protractor the division through which the other side of the angle passes.

To *construct* an angle of given value, draw a straight line, place the protractor so that its diameter coincides with the line, mark the place of the centre of the protractor, and also the point where an arc containing the given number of degrees ends. Then draw a straight line through the points marked.

Protractors are sometimes made in the shape of a rectangle.

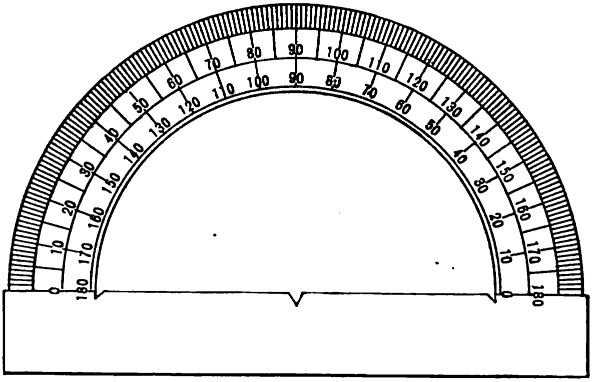


FIG. 72.

EXERCISES

1. Make with a protractor an angle of 30°.

2. Make with a protractor an angle of 45°.

3. Make with a protractor an angle of 100°.

4. Make with a protractor an angle of 150°.

5. Draw an angle and a straight line. Then construct upon the straight line as one side, with the aid of a protractor, an angle equal to the angle you have drawn.

6. Make the angles 30°, 45°, and 60° as well as you can by ruling the lines, but estimating the angle by the eye. Then measure the angles and report the error for each case.

7. Construct an equilateral triangle (see p. 35) and measure its angles. What is the sum of the three angles?

8. Draw a right triangle, measure the two smaller angles, and find their sum.

9. Draw a three-sided figure with unequal sides, measure its angles, and find their sum.

10. Draw a four-sided figure with unequal sides, measure its angles, and find their sum.

11. Draw a four-sided figure with equal sides, measure its angles, and find their sum.

41. Axial Symmetry. If we draw a perpendicular DC from a point D (Fig. 73) to a straight line AB and extend it to D' making CD' equal to CD, the points D and D' are said to be **symmetric points** with respect to the line AB. Suppose that CD is made to revolve about AB through half a revolution. During this motion CD will not change in length, and will remain perpendicular to AB. Therefore, after half a revolution CD will fall upon CD', and D will coincide with D'. Accordingly, D and D' are said to have **axial symmetry** with respect to the line AB, and AB is called the **axis of symmetry.**

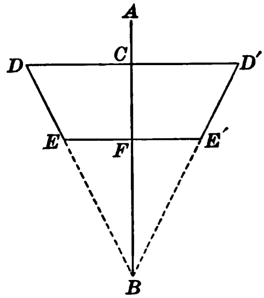


FIG. 73.

The points E and E' are also **symmetric points** with respect to the line AB.

Any two points so situated that the straight line joining them is perpendicular to AB and bisected by AB are symmetric points with respect to AB.

The straight lines DE and $D'E'$ are symmetric lines with respect to the axis AB. By folding DE about AB we can make it coincide with $D'E'$, and thus see that for every point in DE there is a corresponding symmetric point in $D'E'$.

The plane figures $CDEF$ and $CD'E'F$ are symmetric figures with respect to the axis AB. In general,

Two plane figures are said to be symmetric with respect to an axis if every point in the boundary of one has a corresponding symmetric point in the boundary of the other.

Two plane figures symmetric with respect to an axis, as $CDEF$ and $CD'E'F$, are *equal figures;* for either can be made to coincide with the other by revolving it about AB through half a revolution.

The single figure $DEE'D'$ (Fig. 73, p. 47), formed by putting together the equal figures $CDEF$ and $CD'E'F$, is sometimes called a symmetric figure with respect to AB, but a better name for it is a **bi-symmetric** figure, having AB for the axis of symmetry. Therefore, a plane figure is bi-symmetric if it can be divided by a straight line into two equal parts, symmetric with respect to the line of division.

The bi-symmetric form is not confined to geometry. It is seen in the shapes of various common objects, such as a bottle or a lamp shade. It is employed with numerous variations in the decorative arts and in architecture. It is the form chosen by nature for the shape of a leaf.

EXERCISES

1. Name a bi-symmetric four-sided figure in Fig. 73, p. 47.

2. Name two symmetric triangles in Fig. 73.

3. Compare the angles which two symmetrically placed straight lines form with the axis of symmetry. Illustrate by referring to Fig. 73.

4. In Fig. 74 name the axis of symmetry, pairs of corresponding points, pairs of symmetric lines, two symmetric figures, and a bi-symmetric figure.

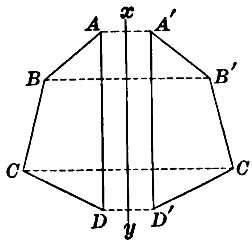


FIG. 74.

5. Make a bi-symmetric figure with paper and scissors.

Draw on the paper a straight line AB (Fig. 75). Fold the paper on AB and sketch lightly the lines AC and CB on the upper fold. Then cut out the figure along the lines AC and CB. Unfold, and the bi-symmetric figure is formed, as seen on the right (Fig. 75).

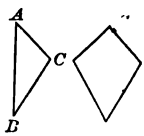


FIG. 75.

6. Make with paper and scissors the bi-symmetric figure shown on the right in Fig. 76.

7. Make in a similar way a bi-symmetric figure according to your own design.

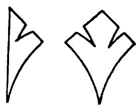


FIG. 76.

8. Must a figure made in this way always be bi-symmetric?

9. Can symmetric figures be bounded by curved lines?

10. Make a cross like that shown in Fig. 77.

Begin by drawing carefully one half the cross as represented in the figure. Take the lengths as follows: AB, 4 in.; AF, ½ in.; FG, GE, and ED, 1 in. Fold on AB as an axis. Prick pin holes through the points F, G, etc. Unfold, and connect the pin holes on the left of AB by straight lines.

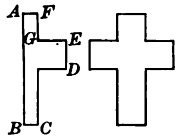


FIG. 77.

11. Make a spool like that shown in Fig. 78 by the pin-hole method.

12. Make the spool in Fig. 78 by the printing method.

First construct the right half of the spool, using a *soft* lead pencil. Moisten the paper with a sponge. Then fold on the axis, lay a book over the paper, and strike it a blow with the fist. Then unfold.

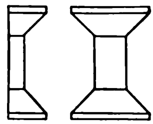


FIG. 78.

13. Devise a way of constructing the bi-symmetric figure in Fig. 77 or in Fig. 78 without folding the paper. Then construct the figure.

14. Construct the bi-symmetric figure suggested in Fig. 79, A.

15. Construct the bi-symmetric figure suggested in Fig. 79, B.

16. Construct the bi-symmetric figure suggested in Fig. 79, C.

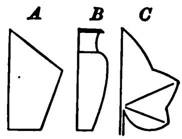


FIG. 79.

17. Is a sorrel leaf (Fig. 80, A) a bi-symmetric figure? Give a reason for your answer. Is the form in Fig. 80, B, bi-symmetric? Construct it.

18. Draw a square, and then draw as many axes of symmetry as possible.

19. Draw a rectangle, and then draw as many axes of symmetry as possible.

20. Draw an equilateral triangle, and then draw as many axes of symmetry as possible.

21. Point out the bi-symmetric objects in the room.

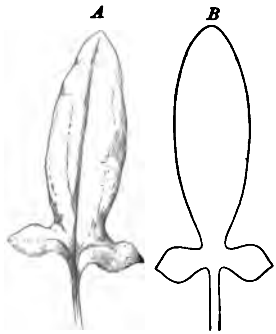


FIG. 80.

42. Central Symmetry. If a plane figure after being turned in its own plane about a point through *half* a revolution coincides with its original position, the figure is said to have **twofold symmetry** with respect to the point; and the point is called the **centre of symmetry.**

An example of twofold symmetry is seen in Fig. 81.

In the figure $ABCD$ the lines AC and BD, intersecting at O, are perpendicular to each other. Also $OA = OC$, and $OB = OD$. Pin tracing paper upon the figure at O, and trace

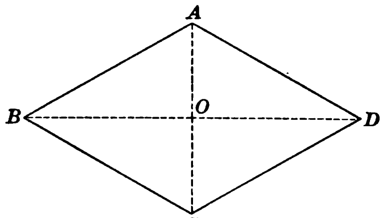


FIG. 81.

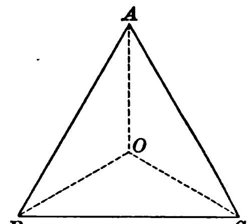


FIG. 82.

its outline. Then turn the tracing paper halfway round the pin as an axis. A and C exchange places; likewise B and D; and the two figures again coincide.

If we regard Fig. 81 as an example of *axial* symmetry, how many axes of symmetry has it? Name them.

If a figure coincides with its original position after being turned *one third* of a revolution in its own plane about a point, the figure is said to have **threefold symmetry,** and so on.

An equilateral triangle (Fig. 82) has threefold symmetry with respect to the intersection O of the perpendiculars drawn from the vertices to the opposite sides, as may be shown by the aid of tracing paper.

If we regard Fig. 82 as an example of *axial* symmetry, how many axes of symmetry has it? Describe their situation.

Nature shows a marked preference for forms having central symmetry. This is seen by examining the petals of flowers (Fig. 83) and the forms of snow crystals (Fig. 85, *F*, *G*, *H*).

FIG. 83.

EXERCISES

1. Make petals similar to those of which the flowers in Fig. 83 are composed, but having perfect central symmetry.

Take thick paper, and describe upon it a circle. Divide the circumference into three equal parts. Draw radii to two of the points of division. Between the radii sketch free-hand one of the petals. Cut out the figure. This figure is a pattern of one of the petals. Place it on the paper. Draw its outline. Pass a pin through the point of the petal and turn the pattern about the pin as a centre until you can again draw its outline beside the first sketch. Repeat, and Fig. 84 will be the result.

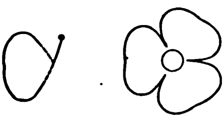


FIG. 84.

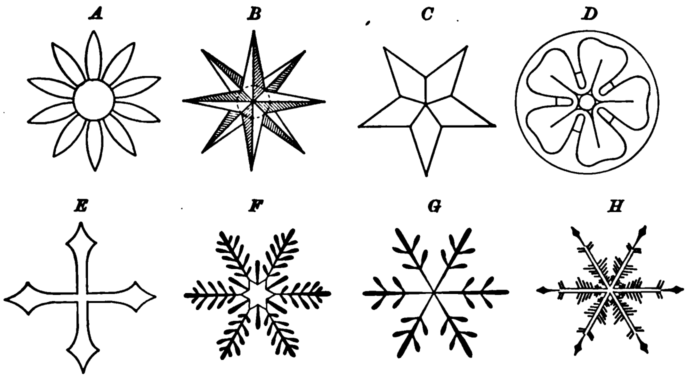


FIG. 85.

2. How many-fold symmetry has each of the figures in Fig. 85?

43. Distance from a Point to a Straight Line. *The shortest line from a point to a straight line is the perpendicular from the point to the line.*

This truth may be established by the aid of axial symmetry as follows:

Let PC (Fig. 86) be a perpendicular drawn from a point P to a straight line AB, and let PD be any other straight line drawn from P to AB. Extend PC to P' making CP' equal to PC, and draw $P'D$. The triangles PDC and $P'DC$ are symmetric with respect to AB. Therefore, they are equal, and $PD = P'D$. Now PP' is less than $PD + P'D$, because a straight line is the shortest line from one point to another. But $PP' = 2\,PC$, and $PD + P'D = 2\,PD$. Therefore, $2\,PC$ is less than $2\,PD$. Hence, it follows, by dividing by 2, that PC is less than PD.

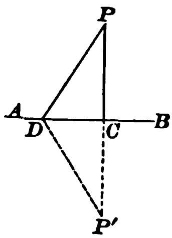


FIG. 86.

This is true however PD is drawn, provided it is not perpendicular to AB. Hence, we conclude that the shortest line from P to AB is the perpendicular PC.

The length of the perpendicular from a point to a straight line is called the **distance from the point to the line.**

44. Parallel Lines further examined. Parallel lines have been defined as straight lines that lie in the same plane and cannot meet however far extended (p. 11).

Now it is easy to imagine two straight lines as extending indefinitely, but our power to observe and represent them is very limited, and practically confined to comparatively short distances. How then can we be sure that parallel lines really exist? This question may be answered by again calling axial symmetry to our aid.

Let AB and CD (Fig. 87) be two straight lines perpendicular to the straight line MN, and intersecting it at the points E and F, respectively. The figures $AEFC$ and $BEFD$ are symmetric with respect to MN as an axis. Hence, if $AEFC$ is revolved about MN, it can be made to coincide with $BEFD$. Therefore, if the lines AB and CD intersect on the left of MN, they must also intersect on the right of MN; that is, they must have *two* points of intersection. Now two straight lines can have only *one* point of intersection. We conclude, therefore, that the two lines cannot meet at all.

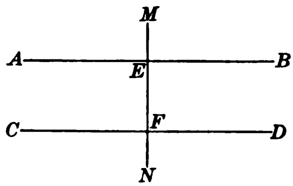


FIG. 87.

Two straight lines in a plane perpendicular to the same straight line are parallel lines.

The line MN is perpendicular to both the parallels AB and CD. It can be proved that through any point on one of two parallels a straight line perpendicular to both can be drawn. The length of this common perpendicular comprised between the parallels is taken as the **distance between the parallels** at that point.

From axial symmetry it follows that

Two parallel straight lines are everywhere equally distant.

Let AB and CD (Fig. 88) be perpendicular to MN, and therefore parallel lines. Let EF be a common perpendicular to AB and CD, and GH be the position taken by EF after it is revolved about MN through half a revolution. From symmetry it follows that $EF = GH$, and this equality holds good wherever on the line AB the point E may be taken.

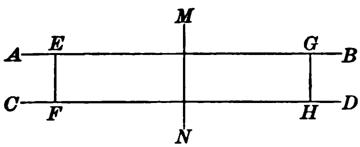


FIG. 88.

45. In Fig. 89 two parallel lines AB and CD are cut by a straight line EF. Let the eight angles thus formed be named by the letters a, b, c, d, e, f, g, h, as indicated.

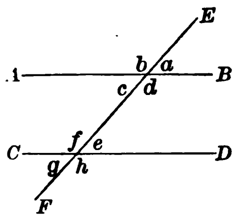


FIG. 89.

The angles a and e, and other pairs similarly placed with respect to AB and CD, are called **exterior-interior** angles.

The two pairs of angles c and e, d and f are called **alternate-interior** angles.

There are two very simple truths which should be remembered about these angles.

1. *The exterior-interior angles of parallel lines are equal.*
2. *The alternate-interior angles of parallel lines are equal.*

EXERCISES

1. Name all the pairs of exterior-interior angles in Fig. 89.
2. Name all the pairs of alternate-interior angles in Fig. 89.
3. If $a = 40°$ (Fig. 89), find the values of the other angles.
4. If $a = 90°$, find the values of all the other angles.
5. If $a = 60°$, find the values of all the other angles.
6. Suppose we draw two parallel lines AB and CD, and then draw a third line EF perpendicular to AB. What position must EF have with respect to CD?
7. Draw two parallel lines. Draw a perpendicular to one of them cutting both parallels. Find the value of each one of the eight angles thus formed.
8. Draw two parallel lines. Cut them by a third line. Then find the values of the eight angles formed, using your protractor as little as possible.
9. What relation exists between the angles d and e in Fig. 89?

One way to discover an answer is to assume d equal to several values, as 50°, 70°, 100°, and in each case find the numerical value of e.

A shorter and better way is to compare d and e by applying one of the truths stated above in italics.

REVIEW EXERCISES

1. What is the locus of a point so moving on a plane surface that it is always at the same distance from a straight line on the surface?

2. The radius of a circle is 5 cm. What is the locus of a point so moving in the plane of the circle that it is always 3 cm distant from the circumference of the circle?

3. Describe a semicircle, draw its diameter, and join any three points in the arc to the ends of the diameter. Measure the angles formed at the three points. Do your results indicate any general truth?

4. A ship sailing due northeast shifts its course to the left through one and a half right angles. In what direction is the ship now sailing?

5. A man walks 2 miles, then turns to his right through a right angle and walks 3 miles; then turns to his left through a right angle and walks 1 mile. Draw a plan and find his distance from his starting point. Take on the paper 1 in. to represent 1 mile.

6. A man walks 6 miles, turns to his left through an angle of 45° and walks 2 miles, then turns through 90° to his left and walks 2 miles. How far in a straight line is he now from his starting point? Take on the paper 1 in. to represent 1 mile.

7. Find the sum of $28° 39'$, $37° 48' 35''$, and $78° 9' 55''$.

8. Find the difference between $48° 5'$ and $37° 27'$.

9. What is the supplement of 5 times $17° 21'$?

10. What is the complement of 3 times $18° 42'$?

11. The angular magnitude about a point is divided into seven equal angles. Find the value of each one correct to the nearest second.

12. The circumference of a circle is divided into seven equal parts. Find the number of degrees, minutes, and seconds in each part.

13. Examine and describe the following objects as regards symmetry:
    (1) a horse;
    (2) an ordinary door;
    (3) a wheel with a dozen spokes;
    (4) a man and his image in a mirror.

14. Draw a bi-symmetric figure bounded by straight lines.

15. Draw a bi-symmetric figure bounded by curved lines.

16. Draw a central-symmetric figure bounded by straight lines.

17. Draw a central-symmetric figure bounded by curved lines.

CHAPTER III

TRIANGLES AND QUADRILATERALS

46. A plane figure bounded by three straight lines is called a **triangle.** The bounding lines are called the **sides**; the points of intersection of the sides are called the **vertices**; the sum of the lengths of the sides is called the **perimeter**; and the angles included by the sides are called the **angles.**

A triangle is called **equilateral,** if all the sides are equal; **isosceles,** if two sides are equal; **scalene,** if no two sides are equal; **acute,** if all the angles are acute; **right,** if one angle is a right angle; **obtuse,** if one angle is an obtuse angle.

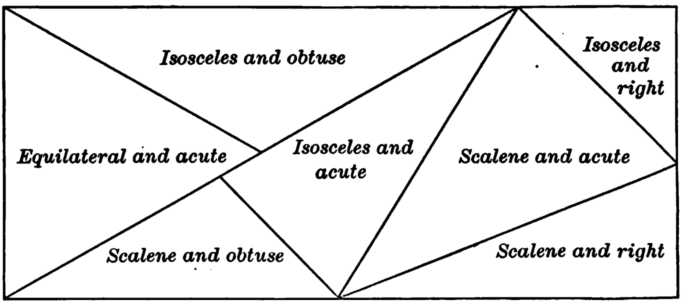


FIG. 90.

The equal sides of an isosceles triangle are called the **legs,** and the third side is called the **base.**

The two sides that include the right angle in a right triangle are called the **legs**; the third side, the **hypotenuse.**

47. The **dimensions** of a triangle are the **base** and the **altitude.** Any side of a triangle may be taken as the base. The **altitude** is the perpendicular from the vertex to the base or to the base produced.

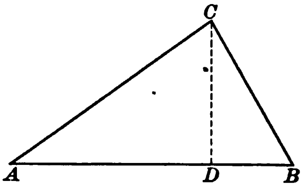


FIG. 91.

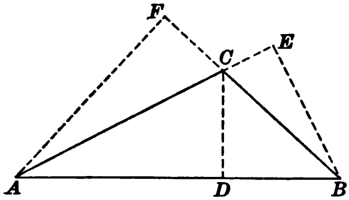


FIG. 92.

In Fig. 91 the base is AB and the altitude is CD.

In the obtuse triangle ABC (Fig. 92) two of the altitudes, AF and BE, lie outside of the triangle.

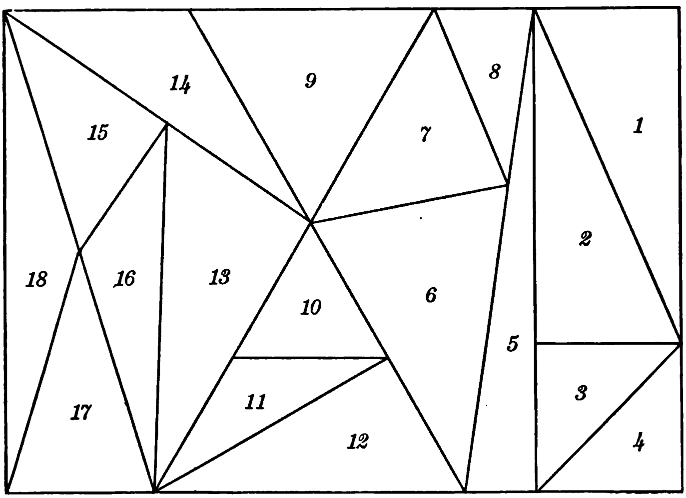


FIG. 93.

Describe the kinds of triangles in Fig. 93.

48. A plane figure bounded by four straight lines is called a **quadrilateral.**

The terms sides, vertices, perimeter, and angles have the same meaning in quadrilaterals as in triangles.

A **diagonal** of a quadrilateral is a straight line that joins two opposite vertices.

A quadrilateral is called a **parallelogram,** if the opposite sides are parallel; a **trapezoid,** if two, and only two, sides are parallel; a **trapezium,** if no two sides are parallel.

A parallelogram is called a **rectangle,** if the angles are all right angles; a **rhomboid,** if the angles are not right angles.

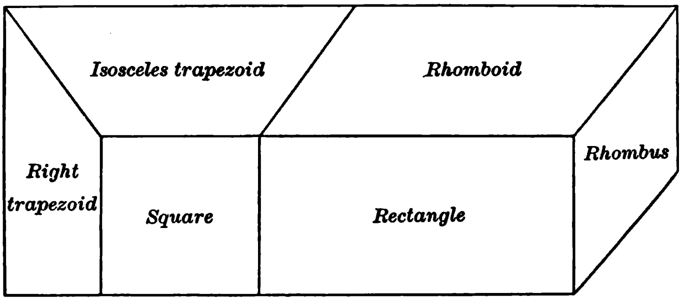


FIG. 94.

A rectangle is called a **square,** if the four sides are equal.

A rhomboid is called a **rhombus,** if the four sides are equal.

The parallel sides of a trapezoid are called the **bases,** and the non-parallel sides are called the **legs.**

A trapezoid is called an **isosceles trapezoid,** if the legs are equal; a **right trapezoid,** if one leg is perpendicular to the bases.

49. Any side of a parallelogram may be taken as the **base.** The **altitude** of a parallelogram is the perpendicular distance from the base to the opposite side.

The line DE is the altitude of $ABCD$ (Fig. 95).

The line DB is a diagonal of the parallelogram $ABCD$.

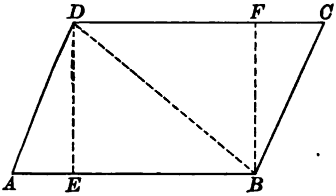


FIG. 95.

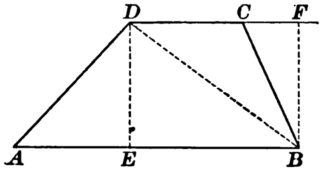


FIG. 96.

The **altitude** of a trapezoid is the perpendicular distance between the bases.

Thus, DE is the altitude of $ABCD$ (Fig. 96).

BD is a diagonal of the trapezoid $ABCD$.

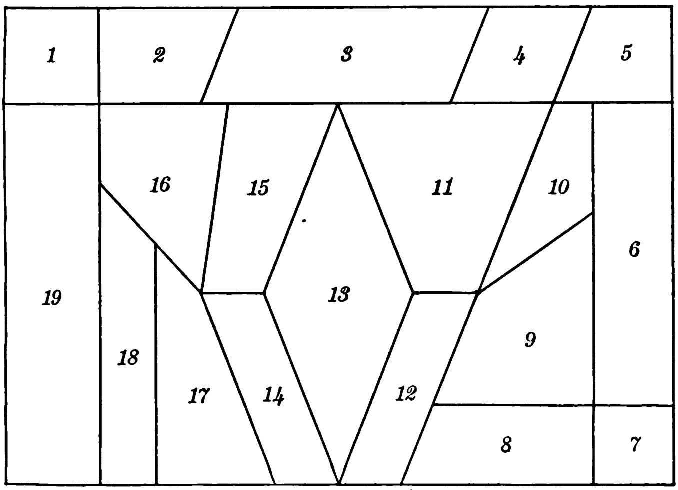


FIG. 97.

Describe the different kinds of quadrilaterals in Fig. 97.

50. Sum of the Angles of a Triangle. When a straight line is made to rotate about one of its points until its direction is opposite to that which it has at the start, the amount of rotation is 180°. With this fact in mind let us find the sum of the angles of a triangle.

Take any triangle ABC (Fig. 98), and let a, b, c denote the values of the angles that have as vertices the points A, B, C, respectively. Place a straight line DE upon the side AC and regard it as having the direction from D to E. Turn DE *clockwise* (as shown by the curved arrows), first about A, through the angle a, until it covers the side AB; then about B, through the angle b, until it covers the side BC; and then about C, through the angle c, until it covers again the side AC. Evidently DE will now have its direction *opposite* to that which it had at first. Hence, it must have revolved through exactly 180°. It is obvious that the whole amount of rotation reckoned in degrees is equal to $a + b + c$. Therefore, $a + b + c = 180°$. We thus arrive at the very useful truth: *The sum of the angles of a triangle is equal to two right angles, or 180°.*

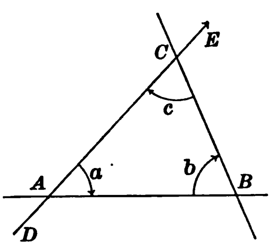


FIG. 98.

EXERCISES

1. How many right angles can a triangle have? How many obtuse angles? How many acute angles? Give your reason in each case.

2. One angle of a triangle is 30°. What is the sum of the other two?

3. Two angles of a triangle are 25° and 75°. Find the third angle.

4. If one acute angle of a right triangle is 40°, what is the value of the other acute angle?

5. What is meant by the statement that one acute angle of a right triangle is the complement of the other?

6. If the acute angles of a right triangle are equal, what is the value of each angle?

7. One angle of a right triangle is 20°. If all the sides of the triangle are extended, find the values of the new angles thus formed.

8. Two angles of a triangle are each 40°. Find the third angle.

9. The values of two angles of a triangle are 46° 37′ and 54° 17′. Find the value of the third angle.

10. Give the name of a right triangle if each acute angle is 45°.

The fact that the sum of the angles of a triangle is just 180° may be shown to the eye as follows:

Draw a triangle *ABC* (Fig. 99), and the altitude *CD*. Cut away the triangle from the paper and fold the corners over the dotted lines shown in the figure until they come together at the point *D*. It will be seen that the three angles of the triangle make at *D* a straight angle, or two right angles.

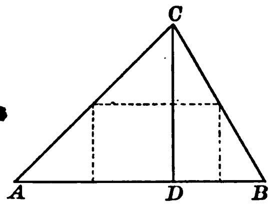


FIG. 99.

51. Sum of the Angles of a Quadrilateral. Take any quadrilateral *ABCD* (Fig. 100). Divide it into two triangles by drawing the diagonal *AC*. The sum of the angles of these two triangles is evidently just the same as the sum of the angles of the quadrilateral. Since the sum of the angles of one triangle is 180°, the sum of the angles of two triangles is $2 \times 180°$, or 360°.

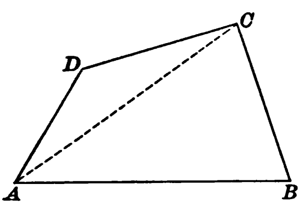


FIG. 100.

The sum of the angles of a quadrilateral is 360°, or four right angles.

EXERCISES

1. If the values of three angles of a quadrilateral are 40°, 70°, and 120°, respectively, what is the value of the fourth angle?

2. If the angles of a quadrilateral are all equal, what is the value of each angle? What kind of a quadrilateral is it?

3. If three of the angles of a quadrilateral are right angles, what is the value of the fourth angle?

4. If two angles of a quadrilateral are supplementary, what is true of the other two angles?

5. Two angles of a quadrilateral are supplementary, and another angle is equal to 66°. What is the value of the fourth angle?

52. Some truths relating to triangles and quadrilaterals will now be stated without proofs. The pupil should know them and apply them before he comes to a logical study of the subject. Each truth, or **theorem** as it is called, should be illustrated by a figure properly drawn and lettered.

THEOREMS

1. *The sum of the acute angles of a right triangle is equal to 90°.*

2. *The middle point of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.*

3. *The angles opposite the equal sides of an isosceles triangle are equal.*

4. *If two angles of a triangle are equal, the sides opposite the equal angles are equal and the triangle is isosceles.*

5. *An isosceles triangle is divided into two equal right triangles by a perpendicular from the vertex to the base.*

6. *The three angles of an equilateral triangle are equal.*

7. *The three altitudes of an equilateral triangle are equal.*

8. *An equilateral triangle is divided into two equal right triangles by each of its altitudes.*

9. *Two triangles are equal:*

(1) *If two angles and the included side of one are equal, respectively, to two angles and the included side of the other.*

(2) *If two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.*

(3) *If the three sides of one are equal, respectively, to the three sides of the other.*

(4) *If the triangles are right triangles and have a leg and the hypotenuse of one equal, respectively, to a leg and the hypotenuse of the other.*

10. *The opposite sides of a parallelogram are equal, and the opposite angles are equal.*

11. *A diagonal divides a parallelogram into two equal triangles.*

12. *The diagonals of a parallelogram bisect each other.*

13. *The diagonals of a rectangle are equal.*

14. *The diagonals of a rhombus bisect each other at right angles.*

15. *The diagonals of a square are equal and also perpendicular to each other.*

16. *The two angles at each base of an isosceles trapezoid are equal.*

EXERCISES

1. How can a right triangle be divided into two isosceles triangles?

2. If an angle at the base of an isosceles triangle is 40°, find the values of the other angles.

3. If the angle at the vertex of an isosceles triangle is 50°, find the values of the other angles.

4. What is the value of each acute angle of an isosceles right triangle?

5. What is the value of each angle of an equilateral triangle?

6. If we divide an equilateral triangle into two equal right triangles by drawing one of the altitudes, what are the values of the acute angles of these right triangles? Compare the length of the shorter leg of one of these triangles with the length of the hypotenuse.

7. An angle A of a parallelogram $ABCD$ is 40°. Find the values of the other angles.

8. Draw the diagonals of a rectangle, and describe the figures into which the rectangle is divided.

9. Draw the diagonals of a rhombus, and describe the figures into which the rhombus is divided.

10. Draw the diagonals of a square, and describe the figures into which the square is divided.

11. Are two triangles equal, if the three angles of one are equal, respectively, to the three angles of the other? Illustrate by figures.

12. Draw an acute triangle and its three altitudes. Do the altitudes intersect in a common point?

13. Draw an obtuse triangle and its three altitudes. Do the altitudes when produced intersect in a common point?

53. Fundamental Problems. The following problems are to be solved with ruler and compasses. Lengths given in centimeters or inches may be laid off with the graduated ruler, but in other cases the dividers should be used.

54. Problem 1. *To construct a straight line equal to a given straight line.*

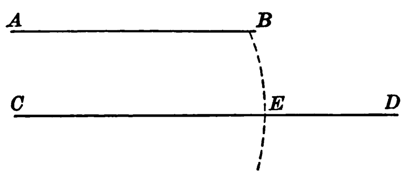


FIG. 101.

Let AB (Fig. 101) be the given straight line.

Draw a straight line CD longer than AB.

With C as centre and radius equal to AB, cut CD at E.

Then CE is the line required.

55. Problem 2. *To bisect a given straight line.*

Let AB (Fig. 102) be the given straight line.

With A and B as centres and with equal radii greater than half of AB, describe arcs intersecting at the points C and D.

Draw CD intersecting AB at M.

Then M is the middle point of AB.

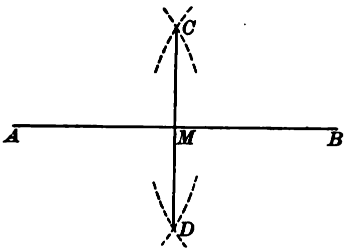


FIG. 102.

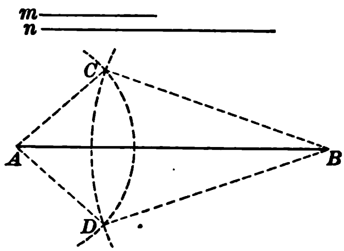


FIG. 103.

56. Problem 3. *To find a point which shall be at given distances from two given points.*

Let A and B (Fig. 103) be the given points, m and n the given distances.

With A as centre and m as radius, describe an arc.

With B as centre and n as radius, describe another arc intersecting the first arc at the points C and D.

The points C and D both satisfy the given condition.

EXERCISES

1. Make a straight line twice as long as AB in Fig. 102, bisect it, and test the accuracy of your work.

2. Draw a triangle, bisect its sides, and join each point of bisection to the opposite vertex of the triangle. Do these lines intersect in a point?

3. Under what condition has Problem 3 no solution? Under what condition has the problem only one solution?

57. Problem 4. *To draw a perpendicular from a given point to a given straight line.*

Let AB (Fig. 104) be the given straight line and C the given point.

With C as centre and a suitable radius, cut AB at H and K.

H and K are equidistant from C.

Find by Problem 3 another point O, equidistant from H and K. Draw CO and produce it to meet AB in M.

Then COM is the perpendicular required.

1. Draw a triangle, and then draw its three altitudes. Do the altitudes intersect in a common point?

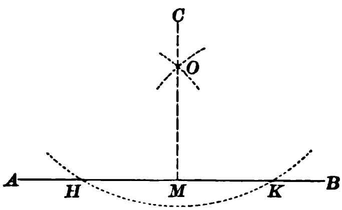


FIG. 104.

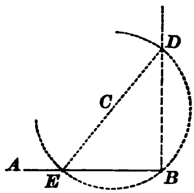


FIG. 105.

58. Problem 5. *To erect a perpendicular at a given point of a given straight line.*

First Method. Proceed as in Problem 4.

Second Method. Let AB (Fig. 105) be the given straight line and B the given point of the line AB.

Take a point C, not in AB; from C as centre and with CB as radius, describe an arc cutting AB at E.

Draw EC and extend it to meet the arc again at D.

Draw BD. Then BD is the perpendicular required.

1. Draw a triangle, and erect perpendiculars at the middle of its sides. Do the three perpendiculars intersect in a common point?

59. Problem 6. *To bisect a given angle.*

Let BAC (Fig. 106) be the given angle.

From centre A and with any convenient radius, describe an arc cutting the sides of the angle in D and E, respectively.

From D and E as centres and with equal radii, describe arcs intersecting at F. Draw AF. Then AF is the bisector required.

1. Draw a triangle, and then bisect its angles. Do the three bisectors intersect in a common point?

2. Bisect a right angle, and bisect one of its halves. How many degrees are there in the angle thus obtained?

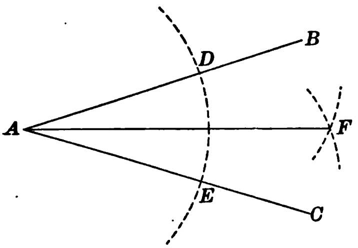


FIG. 106.

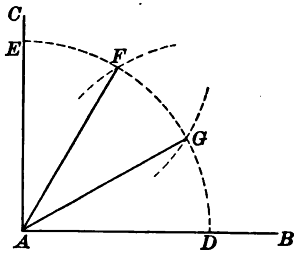


FIG. 107.

60. Problem 7. *To trisect a given right angle.*

Let BAC (Fig. 107) be the given right angle.

From A as centre and with any convenient radius, describe an arc cutting AB at D and AC at E.

From D as centre and with the same radius as before, describe an arc cutting the arc DE at F. Draw AF.

From E as centre and with the same radius, describe an arc cutting the arc DE at G. Draw AG.

Then $DAG = GAF = FAE = \frac{1}{3}\, BAC$.

1. What is the value in degrees of the angle DAG? What is the value in degrees of the angle DAF?

2. Construct an angle of 60°; an angle of 30°.

61. Problem 8. *At a given point in a given straight line to construct an angle equal to a given angle.*

Let AB (Fig. 108) be the given straight line, C the given point, and DEF the given angle.

From E as centre and with any radius, describe an arc cutting the sides of the angle DEF in G and H, respectively.

From C as centre and with the same radius as before, describe an arc LM, cutting AB at L.

From L as centre and with a radius equal to the distance of H from G, cut the arc LM at N.

Draw CN.

Then the angle BCN is the angle required.

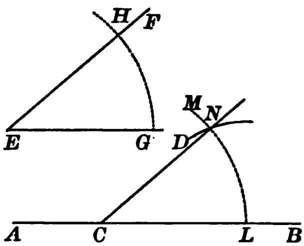


FIG. 108.

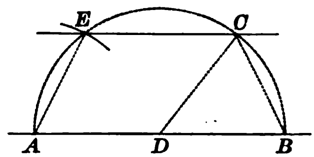


FIG. 109.

62. Problem 9. *To draw through a given point a line parallel to a given straight line.*

Let AB (Fig. 109) be the given straight line and C the given point.

Join C to any point D of AB.

From D as centre and with DC as radius, describe the semicircle $AECB$ and draw BC.

With A as centre and a radius equal to BC, cut the semicircle at E, and draw EC.

Then the line EC is the parallel required.

63. Problem 10. *To divide a given straight line into a given number of equal parts.*

Let AB (Fig. 110) be the given straight line and *three* the given number of parts.

Draw AC, making a convenient acute angle with AB.

Lay off on AC three equal lengths, AD, DE, EF, and draw FB.

Draw straight lines through D and E parallel to FB, meeting AB at the points G and H, respectively.

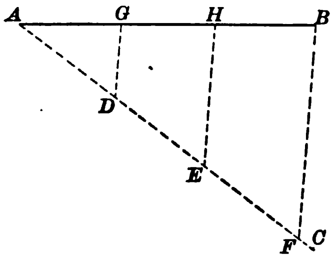


FIG. 110.

Then $AG = GH = HB$. The line AB is said to be trisected.

1. Draw a straight line, and divide it into five equal parts.
2. Draw a straight line, and divide it into four equal parts.

64. Problems of Construction. It now remains to apply the problems just given to the construction of triangles and quadrilaterals. If angles are given in degrees, construct them with the aid of a protractor. Also, if the measurement of angles in degrees is asked for, use the protractor.

REVIEW EXERCISES

1. Construct a right triangle having for legs 8 cm and 4 cm. Measure its acute angles. What is their sum? What ought the sum to be?

2. Construct a right triangle having one leg 6 cm long, and the acute angle adjacent to this leg equal to 50°.

3. Construct a right triangle with the hypotenuse 10 cm long, and an acute angle equal to 35°.

4. Construct a right triangle with one leg 4 cm long, and the hypotenuse 9 cm long.

5. Construct an isosceles triangle, having given the base equal to 5 cm, and an angle at the base equal to 75°.

6. Construct an isosceles triangle, having given one leg equal to 7 cm, and an angle at the base equal to 40°.

7. Construct an isosceles triangle, having given the base equal to 8 cm, and the angle opposite the base equal to 120°.

Hint. Find the value of an angle at the base.

8. Construct an isosceles triangle, having given the base equal to 6 cm, and one leg equal to 9 cm.

Hint. Erect a perpendicular at the middle of the base.

9. Construct an isosceles right triangle with legs 8 cm long.

10. Construct an isosceles right triangle with a hypotenuse 8 cm long.

Hint. The middle point of the hypotenuse is equally distant from the three vertices of the triangle.

11. Construct an equilateral triangle having a side 6 cm long.

12. Construct an equilateral triangle having an altitude of 6 cm.

13. Construct a triangle having two sides 8 cm and 5 cm, and the included angle 50°.

14. Construct a triangle having angles 40° and 60°, and the included side 7 cm long. What must the third angle be?

15. Draw with the ruler a straight line and two acute angles. Then construct a triangle having the acute angles for the angles and the straight line for the included side.

16. Construct a triangle having for sides 6 cm, 8 cm, and 4 cm.

17. Construct a triangle having for sides 6 cm, 8 cm, and 10 cm. Find by measurement the values of the three angles of the triangle.

18. Draw three unequal straight lines. Then construct a triangle having these lines as sides.

19. Construct a square having a side 6 cm long.

20. Construct a square having a diagonal 10 cm long.

21. Draw a straight line AB. Then construct a square having its perimeter equal to AB.

22. Construct a rectangle with adjacent sides 8 cm and 5 cm.

23. Construct a rectangle having one side 6 cm long, and a diagonal 10 cm long. What is the length of the other side?

24. Construct a rectangle having one side 6 cm long, and a diagonal 12 cm long. Find the angle formed by the diagonals.

25. Construct a rhombus, given one side 6 cm, and one angle 40°.

26. Construct a rhombus, given one side 10 cm, and one diagonal 8 cm.

27. Construct a rhombus having one side equal to one of the diagonals. Into what kind of figures does this diagonal divide the rhombus?

28. Construct a rhomboid having its adjacent sides 9 cm and 5 cm, and the included angle 50°.

29. Construct a parallelogram having its diagonals 7 cm long and 12 cm long, and intersecting so as to form an angle of 55°.

30. Construct a parallelogram having a base 8 cm long, and the altitude equal to 6 cm. Is more than one solution of this problem possible?

31. Construct an isosceles trapezoid, given one base 10 cm, one leg 7 cm, and the angle between them 60°.

32. Construct a right trapezoid, given the bases 9 cm and 5 cm, and the leg inclined to the bases 10 cm.

33. Construct a quadrilateral having two adjacent sides each equal to 8 cm, the other two sides each equal to 4 cm, and the angle formed by the first two sides equal to 35°. What kind of a quadrilateral is it?

34. Make a plan of two vertical posts 18 ft. high and 24 ft. apart, with a horizontal cross-bar 10 ft. from the ground. Represent one foot by a line 0.5 cm long.

35. Make a plan of a wall 20 ft. high and 64 ft. long with a door through the middle 10 ft. high and 16 ft. wide. Represent 1 ft. by a line 0.2 cm long.

36. A stone dam is 20 ft. high, 34 ft. wide at the bottom, and 4 ft. wide at the top. The slant height on each side is the same. Draw to scale a cross-section of the dam, and find by measurement the slant height. What kind of a plane figure is the cross-section?

CHAPTER IV

CIRCLES AND REGULAR POLYGONS

65. Definitions. A **central angle** is an angle formed by two radii.

A **sector** is the portion of a circle bounded by two radii and the arc contained between them.

A sector whose central angle is a right angle is called a **quadrant.**

A **segment** is the portion of a circle bounded by an arc and its chord.

EXERCISES

1. Name in Fig. 111 a central angle, a sector, a quadrant, and a segment.

2. Into how many quadrants can a circle be divided?

3. Is a semicircle a sector or a segment?

4. Into what figures is a circle divided by drawing a chord?

5. Into what figures is a sector divided by the chord of its arc?

6. Construct a quadrant.

7. Construct a sector whose angle is 45°.

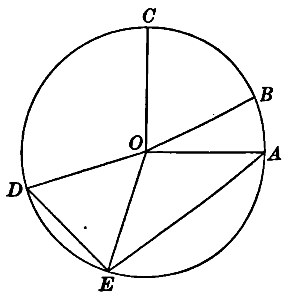


FIG. 111.

8. Construct sectors of 30° and 120°, and compare them as regards magnitude.

9. Construct a segment whose chord is equal to the radius of the circle. What kind of a triangle is formed by drawing radii to the ends of the chord?

66. Radius and Chord. A sector AOB (Fig. 112) is a bi-symmetric figure with respect to the radius OC drawn perpendicular to the chord AB.

THEOREMS

1. *A radius perpendicular to a chord bisects the chord and the arc subtended by it.*

2. *A radius that bisects a chord is perpendicular to it.*

3. *A perpendicular erected at the middle point of a chord passes through the centre of the circle.*

Every theorem contains an **hypothesis** or statement assumed to be true, and a **conclusion** or statement to be proved to be true. Thus, in the first of the theorems just stated the hypothesis is that the radius OC is perpendicular to the chord AB, and the conclusions are that $AD = BD$, and the arc $AC =$ the arc BC.

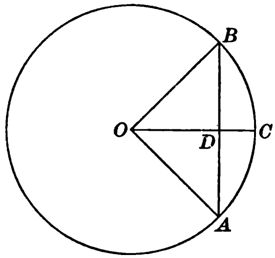


FIG. 112.

EXERCISES

1. What is the hypothesis and what is the conclusion in the second of the theorems above stated?

2. What is the hypothesis and what is the conclusion in the third theorem?

3. If in a circle perpendiculars are erected at the middle points of two chords not parallel, where must be their point of intersection?

4. Describe a circle passing through three given points. For what position of the points is this problem impossible?

5. How many circles can be described through two given points?

6. What is the locus of the centres of all circles that can be described through two given points?

67. Arc and Chord. In the sector AOB (Fig. 113) draw OC perpendicular to AB. Then OC bisects AB (p. 73), and measures the distance of AB from O (p. 52). Let the sector AOB be turned in the plane of the paper about O to a new position, DOE. Since the sector does not change in size or in shape, the arcs DE and AB are equal, the chords DE and AB are equal, and the perpendiculars OF and OC are equal.

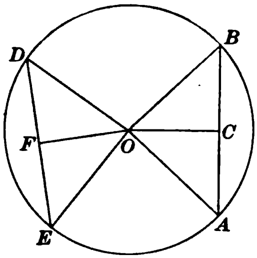


FIG. 113.

THEOREMS

1. *Equal arcs are subtended by equal chords.*
2. *Equal chords subtend equal arcs.*
3. *Equal chords are equally distant from the centre.*

EXERCISES

1. State the hypothesis and the conclusion in each of the above theorems.

2. If angle DOE = angle AOB, what is to be inferred respecting the corresponding arcs and chords?

3. Draw an arc, and then construct an equal arc.

4. Draw an arc, and then bisect it.

5. Upon a given straight line as chord construct an arc of 60°.

6. If the arc AB (Fig. 113) is made to slide round the circumference of the circle, what will be the locus of the middle point C of its chord AB?

7. Describe a circle, and then draw through a point within the circle the longest and the shortest chords that can be drawn.

8. Describe a circle, and then draw through a point within the circle two chords equal in length.

9. Of two unequal chords, which is nearer to the centre?

68. Inscribed Angles. An **inscribed angle** is an angle whose vertex is on the circumference of a circle, and whose sides are chords.

The angle BAC (Fig. 114) is an inscribed angle. The chord BC divides the circle into two segments, and the angle BAC is said to be inscribed in the segment $BADC$.

THEOREMS

1. *An inscribed angle is measured by half the arc intercepted between its sides.*

2. *An angle inscribed in a semicircle is a right angle.*

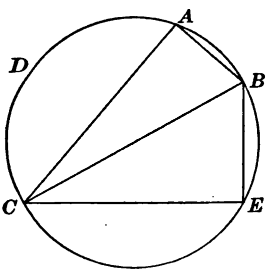


FIG. 114.

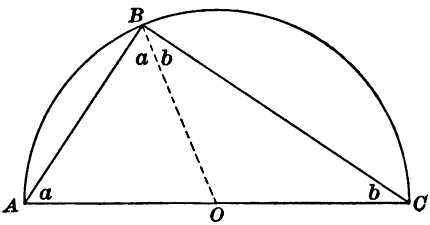


FIG. 115.

Theorem 2 may be proved as follows:

Inscribe an angle ABC (Fig. 115) in a semicircle, and draw the radius OB. The triangles OBA and OBC are isosceles. Therefore, the two angles marked a are equal, and the two angles marked b are equal (p. 62).

The sum of the angles of the triangle $ABC = 180°$.

In other words, $2a + 2b = 180°$. That is, $a + b = 90°$.

But $a + b$ is equal to the inscribed angle ABC.

Therefore, $ABC = 90°$.

EXERCISES

1. What follows from the first theorem on page 75 as regards all angles inscribed in the same segment? Illustrate by a figure.

2. Draw a circle and divide it into two unequal segments. Inscribe an angle in each segment. Is the angle inscribed in the greater segment acute or obtuse? Is the angle inscribed in the smaller segment acute or obtuse? What reasons for your answers can you give?

3. Draw a circle and divide it into two unequal segments. Inscribe an angle in each segment. Now, it is a fact that the sum of these two angles is exactly 180°. Can you discover by the aid of the first theorem on page 75 the reason why this is true?

4. Construct a right triangle having given the hypotenuse = 10 cm, one leg = 6 cm.

Draw a straight line 10 cm long, and upon this line as a diameter describe a semicircle, then with one end of the line as a centre and with a radius equal to 6 cm in length describe an arc cutting the semi-circumference. The point of intersection is the vertex of the right angle of the triangle required.

5. Construct a right triangle having given the hypotenuse = 10 cm, one acute angle = 30°.

6. Construct a right triangle having given the hypotenuse = 10 cm, the altitude upon the hypotenuse as base = 3 cm. What is the greatest value that the altitude could have in this problem?

7. Draw an isosceles right triangle with a hypotenuse 10 cm long.

8. Construct a right angle whose sides pass through two given points. Is there more than one solution?

9. Two straight railroads on a level plain meet at a point A (Fig. 116) and form an angle of 30°. On one of them there is a station B, 3 miles from A and another station C, 11 miles from A. Find a point D, on the other line, such that the angle BDC shall be a right angle. Is there more than one such point? Show that the problem could not be solved if AC were 7 miles. Take 1 cm to represent 1 mile.

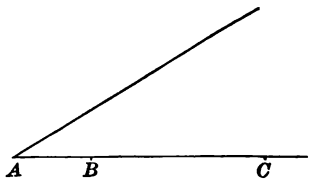


FIG. 116.

69. Secant and Tangent. A straight line that cuts a circumference in two points is called a **secant.**

A straight line that does not cut a circumference, but has one point in common with it, is called a **tangent.** The common point is called the **point of contact.**

In Fig. 117, *ABC* is a secant and *AT* is a tangent. The point *T* is the point of contact. If we imagine the secant *ABC* to turn about *A* till the points *B* and *C* coincide, the secant will become a tangent.

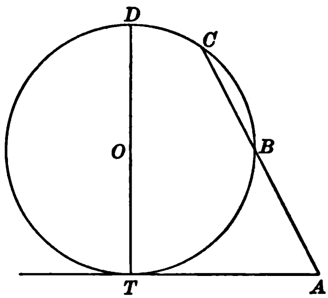


FIG. 117.

A tangent and a circumference are said to touch each other at the point of contact. Evidently all points of a tangent, except the point of contact, lie outside the circumference.

THEOREMS

1. *A tangent is perpendicular to the radius drawn to the point of contact.*

2. *The perpendicular to a radius at its extremity is a tangent to the circle.*

3. *The perpendicular to a tangent at the point of contact passes through the centre of the circle.*

EXERCISES

1. State the hypothesis and the conclusion in each of the Theorems 1, 2, and 3.

2. Describe a circle, and then draw through a point on the circumference a tangent to the circle.

3. If tangents are drawn through the ends of a diameter of a circle, what is their position with respect to each other?

70. Two Tangents. The figure $PAOB$ (Fig. 118) formed by two tangents PA, PB, drawn from an exterior point P to the circle whose centre is O, and the radii OA, OB, drawn to the points of contact, is a bi-symmetric figure with respect to the straight line PO, which joins P to the centre.

Hence, $PA = PB$, and PO bisects the angle APB.

THEOREMS

1. *Two tangents drawn from an exterior point to a circle are equal.*

2. *Two tangents from an exterior point make equal angles with the straight line which joins the exterior point to the centre of the circle.*

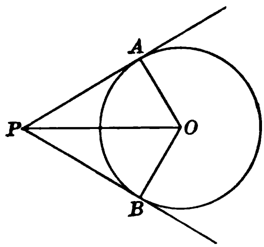


FIG. 118

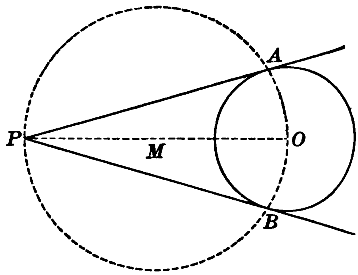


FIG. 119.

EXERCISES

1. What are the values of the angles PAO and PBO (Fig. 118)?
2. What is the sum of the angles APO and AOP?
3. If $APB = 90°$, what kind of a figure is $APBO$?
4. Describe a circle, and then draw tangents to the circle through a point P outside the circumference (Fig. 119).

Upon PO as diameter, describe a circle cutting the given circle at the points A and B. Draw the straight lines PA and PB.

5. Draw two tangents to a circle making an angle of 60°.

71. Railroad Curves. In laying out a railroad curve, care must be taken to avoid a sudden change of direction at the points where the curve begins and ends. At these points, therefore, the straight track must be so joined to the curve that it is a tangent to the curve. How the connection may be made is illustrated in Fig. 120.

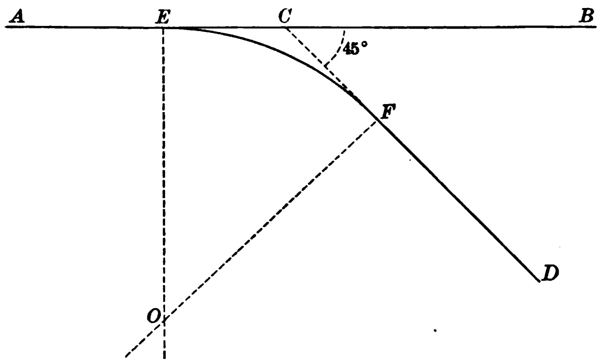


FIG. 120.

A railroad *AB* runs due east, and it is desired to connect with it a branch line *CD* which runs southeast. In this case the total change of direction is 45°. Starting from the point *C*, the engineer lays off equal distances *CE* and *CF* along the lines, taking into account the fact that the greater these distances are, the less rapid will be the change in direction along the curve. At the points *E* and *F*, perpendiculars are erected meeting at the point *O*. If now with *O* as centre and *OE* as radius, an arc *EF* is laid out, it is clear that the straight lines *AE* and *DF* will be tangents to this arc. Thus, a sudden change of direction at the points *E* and *F* is avoided.

72. Two Circles. The relative position of any two circles is determined by the distance between their centres as compared with the sum of their radii. Let d denote the distance between their centres, r and r' their radii.

There are six cases to be considered.

Case 1. d greater than $r + r'$ (Fig. 121). In this case the two circles lie wholly outside each other.

Case 2. $d = r + r'$ (Fig. 122). In this case the circles

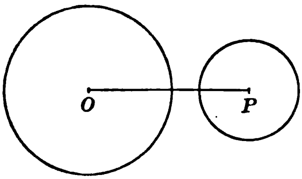


FIG. 121.

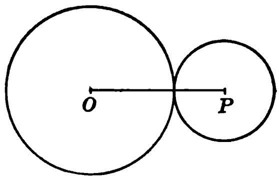


FIG. 122.

touch each other externally, and a tangent common to both circles can be drawn through the point of contact.

Case 3. d less than $r + r'$, but greater than $r - r'$ (Fig. 123). In this case the circles intersect in two points.

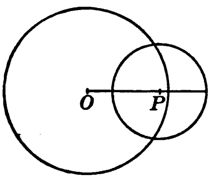


FIG. 123.

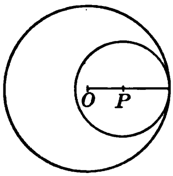


FIG. 124.

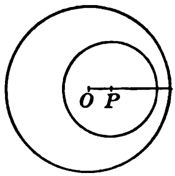


FIG. 125.

Case 4. $d = r - r'$ (Fig. 124). The circles touch internally, and a tangent common to both circles can be drawn through the point of contact.

Case 5. d less than $r-r'$ (Fig. 125). In this case the smaller circle lies wholly within the larger.

Case 6. If $d=0$, the two circles are concentric.

EXERCISES

How many common tangents can be drawn to two circles:

1. When the circles lie wholly outside each other?
2. When the circles touch each other externally?
3. When the circles intersect each other?
4. When the circles touch each other internally?
5. When one circle lies wholly within the other circle?

73. Ornamental Curves. By joining arcs of circles, various useful ornamental curves may be constructed. The following are some examples.

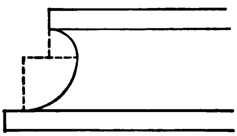


FIG. 126.

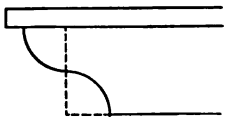


FIG. 127.

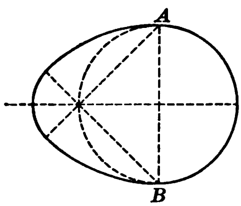


FIG. 128.

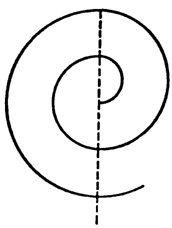


FIG. 129.

EXERCISES

1. Construct curves like those in Figs. 126 and 127.
2. Construct an oval like that in Fig. 128.
3. Construct a spiral like that in Fig. 129.

74. Polygons. A plane figure bounded by straight lines is called **a polygon.** This definition includes the triangle and the quadrilateral, but the term "polygon" is usually applied to plane figures having more than four sides.

Polygons have special names according to the number of their sides. A **pentagon** has 5 sides, a **hexagon** has 6 sides, an **octagon** has 8 sides, a **decagon** has 10 sides, a **dodecagon** has 12 sides, an **icosagon** has 20 sides.

The **sides,** the **perimeter,** and the **vertices** of a polygon are defined exactly as in the case of a triangle.

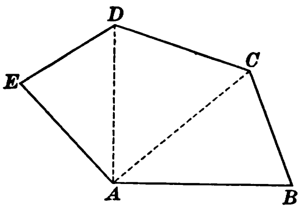


Fig. 130.

A **diagonal** of a polygon is a straight line joining two vertices not adjacent.

By drawing all the diagonals possible from one vertex of a polygon, we can divide the polygon into triangles; the number of triangles thus obtained is always *two less* than the number of sides.

The sum of the angles of the pentagon in Fig. 130 is evidently equal to the sum of the angles of the triangles into which it is divided; that is, equal to $3 \times 180°$.

The sum of the angles of a polygon is equal to 180° multiplied by the number of sides minus two.

EXERCISES

1. Find the sum of the angles of a hexagon.
2. Find the sum of the angles of an octagon.
3. Find the sum of the angles of a decagon.
4. Find the sum of the angles of a dodecagon.
5. Draw a hexagon, and then draw all the diagonals that can be drawn. How many diagonals are there?

75. Regular Polygons. A **regular polygon** is a polygon that has all its sides equal and all its angles equal.

There exists in every regular polygon a point equidistant from all the vertices, and also equidistant from all the sides. This point is called the **centre** of the polygon.

An **angle at the centre** of a regular polygon is the angle formed by two straight lines drawn from the centre to the ends of one side. The angles at the centre of a regular polygon are equal.

Straight lines from the centre of a regular polygon to all the vertices divide the polygon into as many equal isosceles

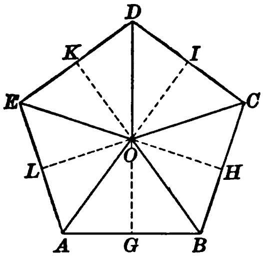


FIG. 131.

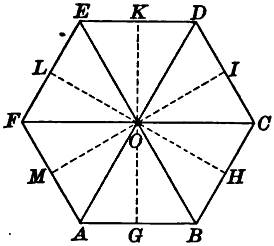


FIG. 132.

triangles as there are sides; and by drawing perpendiculars from the centre to the sides, the polygon is divided into twice as many equal right triangles as there are sides. In this way a regular pentagon (Fig. 131) is divided into five equal isosceles triangles and ten equal right triangles.

A regular polygon has central symmetry. The centre of the polygon is the centre of symmetry. The symmetry is as many fold as there are sides.

Regular polygons also have axial symmetry. A regular hexagon (Fig. 132) has six axes of symmetry.

EXERCISES

1. How can the angle at the centre of a regular polygon be found?

2. Find the values of the angles at the centre of regular polygons having 3, 4, 5, 6, 8, 10, and 100 sides.

3. Find the value of an angle of a regular hexagon.

First find the sum of all the angles of the hexagon (see p. 82), and then divide the sum by 6.

4. Find the value of an angle of a regular octagon.

5. Find the value of an angle of a regular decagon.

6. How can the centre of a given regular polygon be found?

Two methods are suggested by Figs. 131 and 132.

76. Inscribed and Circumscribed Figures. A polygon is said to be **inscribed in a circle** if its sides are chords of the circle, and the circle is said to be **circumscribed about the polygon.**

A polygon is said to be **circumscribed about a circle** if its sides are tangents to the circle, and the circle is said to be **inscribed in the polygon** (Figs. 133 and 134).

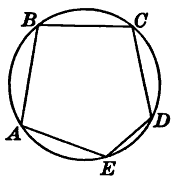


FIG. 133.

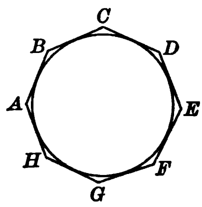


FIG. 134.

In the case of a regular polygon, the centre of the polygon coincides with the centres of the inscribed and the circumscribed circles. The easiest way to construct a regular polygon is to describe a circumference, divide it into as many equal arcs as there are sides in the polygon, and draw the chords of these arcs.

EXERCISES

1. Circumscribe a circle about a given rectangle (Fig. 135).

To solve this problem a point must be found equidistant from the four vertices of the rectangle.

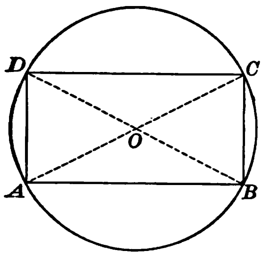


FIG. 135.

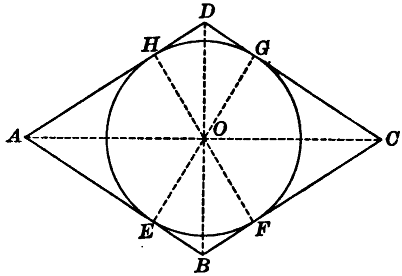


FIG. 136.

2. Inscribe a circle in a given rhombus (Fig. 136).

The diagonals bisect the angles of the rhombus, and the point O where the diagonals intersect is the centre of the inscribed circle.

3. Draw an isosceles trapezoid, and inscribe a circle in it.

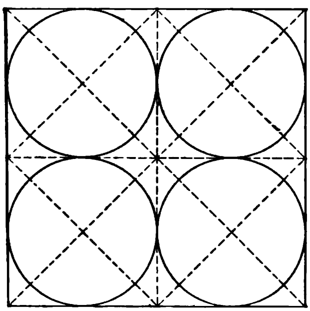


FIG. 137.

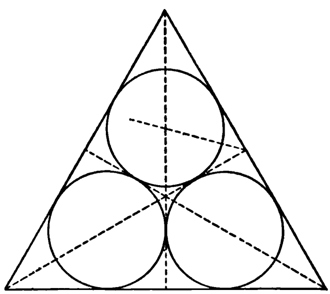


FIG. 138.

4. Construct Fig. 137. Begin by drawing the large square.

5. Construct Fig. 138. Begin by drawing the equilateral triangle.

77. Division of a Circumference. The problem of constructing a regular polygon depends for its solution on our ability to divide a circumference into equal arcs. The theorems of elementary geometry enable us to make this division with the aid of ruler and compasses in three general cases.

Case 1. Division into 2, 4, 8, 16, 32, etc., equal arcs.

A circumference is divided into two equal arcs by simply drawing a diameter, and into four equal arcs by drawing two diameters perpendicular to each other. If these arcs are bisected, the circumference will be divided into eight equal arcs. In practice we bisect one of the four arcs, and lay off the chord of half the arc eight times round the circumference, as shown in Fig. 139.

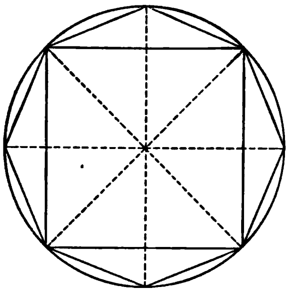


FIG. 139.

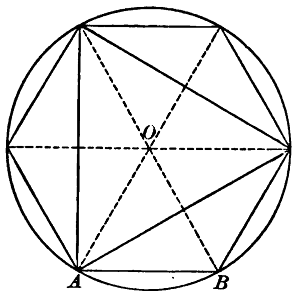


FIG. 140.

Case 2. Division into 3, 6, 12, 24, 48, etc., equal arcs.

Draw a radius OA (Fig. 140). Lay off a chord $AB = OA$. Draw OB. The triangle AOB is equilateral, and therefore the angle $AOB = 60°$. Since $60°$ is contained in $360°$ just six times, it follows that the radius of a circle carried as a chord round the circumference will divide it into six equal arcs.

The first, third, and fifth points trisect the circumference.

Case 3. Division into 5, 10, 20, 40, etc., equal arcs.

Draw two perpendicular diameters *AB* and *CD* (Fig. 141). Bisect the radius *OA* at *E*, and with *E* as centre and the distance *EC* as radius, describe an arc cutting *OB* at *F*. *CF* applied as a chord five times will divide the circumference into five equal arcs, and *OF* applied as a chord ten times will divide it into ten equal arcs.

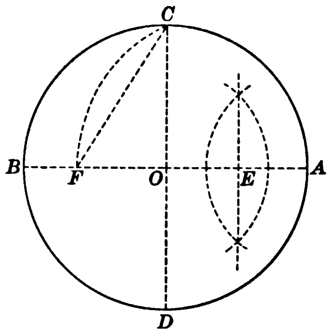


FIG. 141.

The proof involves reasoning of too difficult a character to be given here.

EXERCISES

1. Inscribe a square in a circle.
2. Inscribe a regular octagon in a circle.
3. Circumscribe a square about a circle.
4. Circumscribe a regular octagon about a circle.
5. Inscribe a regular hexagon in a circle.
6. Inscribe an equilateral triangle in a circle.
7. Circumscribe a regular hexagon about a circle.
8. Circumscribe an equilateral triangle about a circle.
9. Inscribe an equilateral triangle in a circle, and circumscribe an equilateral triangle about the same circle. Are the four triangles thus formed equal triangles?
10. Inscribe a square in a circle, and circumscribe a square about the same circle. Draw the diagonals of the inscribed square.
11. Inscribe a regular hexagon in a circle, and circumscribe an equilateral triangle about the same circle.
12. Inscribe a regular pentagon in a circle.
13. Circumscribe a regular decagon about a circle.

78. Stone pavements illustrate the use of regular polygons. Not every regular polygon can be used; only those can be used whose sides touch at all points, for, otherwise, the surface would not be everywhere covered.

FIG. 142.

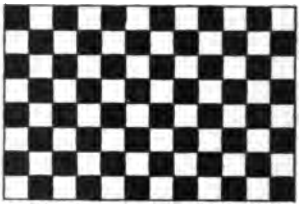


FIG. 143.

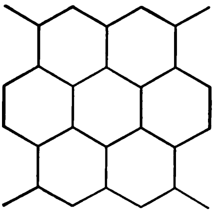


FIG. 144.

Equilateral triangles can be used (Fig. 142) in groups of six about a point; for each angle is 60°, and six of these angles fill 360°, the whole angular magnitude about a point. The arrangement in squares is shown in Fig. 143. Regular hexagons may be arranged in groups of three about a point (Fig. 144).

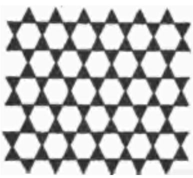


FIG. 145.

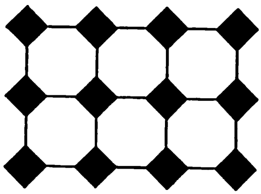


FIG. 146.

In Fig. 145 regular hexagons and equilateral triangles are combined.

In Fig. 146 regular octagons and squares are combined.

79. Problems of Construction. We add a few more examples of the numerous ways in which straight lines and arcs of circles may be combined so as to produce ornamental figures.

1. Construct a regular hexagon. Then, taking as centres the centre of the hexagon and its six vertices, describe circles, each with a radius equal to half of one side of the hexagon. Lastly, describe a circle which shall enclose the whole figure and touch six of the smaller circles.

2. Construct a square. Draw its diagonals. With the intersection of the diagonals as centre and with a radius equal to one fourth of a diagonal, describe a circle. With the vertices of the square as centres and with the same radius as before, describe arcs within the square and bounded by its sides. Join the ends of the four arcs by straight lines. In this way a regular octagon is formed which may be said to be inscribed in the square.

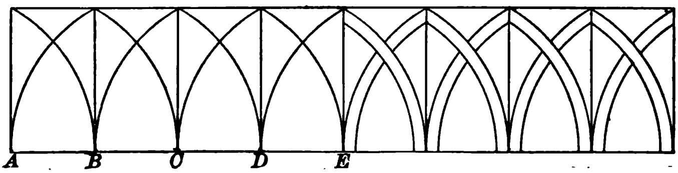


FIG. 147.

3. Construct Fig. 147. The distances AB, BC are equal. The arcs described with A, B, etc., as centres have equal radii. The curved forms produced in this fashion are characteristic of Gothic architecture.

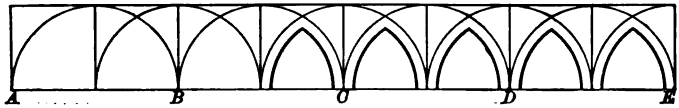


FIG. 148.

4. Construct Fig. 148. The distances AB, BC, etc., are equal to twice the distance between the two horizontal parallels.

5. Construct a five-starred figure (Fig. 149). Begin by describing a circle and marking the vertices of an inscribed regular pentagon.

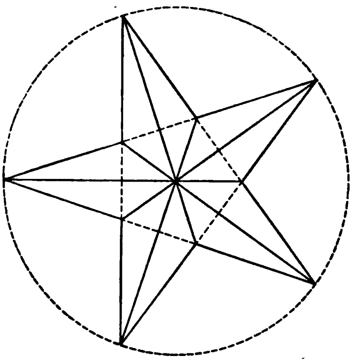


FIG. 149.

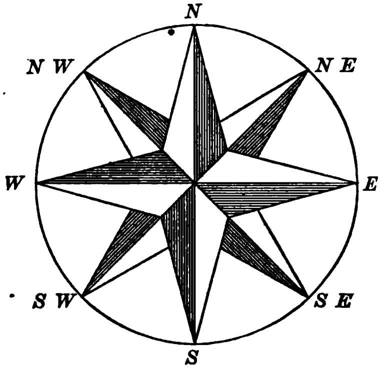


FIG. 150.

6. Construct a compass face with eight rays (Fig. 150). Describe a circle and mark the vertices of an inscribed regular octagon.

7. Construct the design shown in Fig. 151. Describe first the large circle. Each arc within the circle has the vertex of the inscribed regular octagon for centre and a side of the octagon for radius.

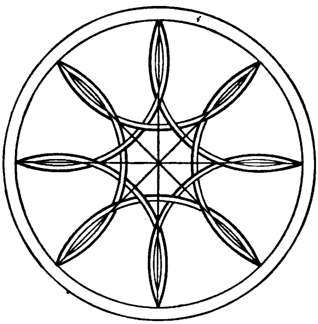


FIG. 151.

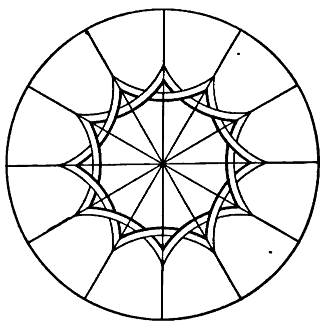


FIG. 152.

8. Construct the ornamental design shown in Fig. 152.

80. Length of a Circumference. The length of the circumference of a circle is greater than the perimeter of an inscribed regular polygon and less than the perimeter of a circumscribed regular polygon. For example, the perimeter of an inscribed regular hexagon is six times the length of the radius of the circle or three times the length of its diameter, but the perimeter of the circumscribed square is four times the length of the diameter. Therefore, the length of the circumference must be more than three times, but less than four times, the length of the diameter.

To find very nearly the relation between the circumference of a circle and its diameter in respect to length, geometers take advantage of the fact that the greater the number of sides of an inscribed or a circumscribed regular polygon, the less the difference between the length of its perimeter and the length of the circumference. They suppose regular polygons of a great number of sides inscribed in the circle and circumscribed around the circle, and then compare the lengths of their perimeters with the diameter of the circle.

In this way it can be proved that the length of a circumference is more than 3.141592 times, but less than 3.141593 times, the length of the diameter. If then we multiply the diameter by 3.14159, we shall obtain very nearly the length of the circumference. In fact, the error made by using this value to compute the circumference of a circle whose diameter is 1 mile is less than 1 inch.

The approximate value usually taken for the ratio of the circumference of a circle to the diameter is 3.1416, or $3\frac{1}{7}$. The true value of the ratio cannot be expressed exactly in figures but is represented by the Greek letter π (Pi).

It follows that *the circumference of a circle is equal to π times the diameter.*

REVIEW EXERCISES

With English units take $\pi = \frac{22}{7}$; *with metric units take* $\pi = 3.1416$.

1. Find the circumference of a circle if the diameter is 35 ft.

2. Find the circumference of a circle if the radius is 14 in.

3. Find the circumference of a circle if the radius is 8 cm.

4. Find the diameter of a circle if the circumference is 132 ft.

5. Find the radius of a circle if the circumference is 1 m.

6. If the driving wheel of a locomotive has a radius of 3 ft., how many revolutions will it make in going 236 miles?

7. What is the diameter of a circular pond if a man take 840 paces of $2\frac{1}{2}$ ft. each in walking around the pond?

8. What should be the diameter of a round dining table for eight persons, allowing 3 ft. for each person?

9. The earth turns on its axis once every 24 hours. Taking its diameter as 8000 miles, how far does a point on the equator move in 1 second?

10. The circumferences of two concentric circles are $16\frac{1}{2}$ ft. and 18 ft. Find the width of the ring between them.

11. The radius of a circle is 7 ft. What is the length of an arc of 30°, or $\frac{1}{12}$ of the circumference?

12. If the radius of a circle is 14 in., find the length of an arc of 45°.

13. The diameter of a circle is 5 ft. 10 in. and the angle formed by two radii is 150°. Find the length of the included arc.

14. The radius of a circle is 28 in. Find the value of the central angle that will subtend an arc 11 in. long.

Solution. Circumference $= \frac{22}{7} \times 2 \times 28$ in. $= 176$ in., and $\frac{11}{176} = \frac{1}{16}$. Therefore, the central angle $= \frac{1}{16}$ of $360° = 22°\ 30'$.

15. The radius of a circle is 8 ft. 2 in. and the length of an arc is 6 ft. 5 in. Find the central angle that subtends this arc.

16. A man, walking 4 miles per hour, walks for 15 minutes on the circumference of a circle, the radius of which is 1 mile. Find the length of the arc passed over and the number of degrees in it.

17. How many degrees, etc., are there in an arc equal in length to the radius of the circle?

CHAPTER V

AREAS

81. Equivalent figures are figures that have the same size.

Equal figures agree both in size and in shape, but two *equivalent figures may have very different shapes.*

For example, the square and the triangle in Fig. 153 are quite unlike in shape, yet they have been so drawn that they enclose equal amounts of surface. Therefore, they are equivalent figures.

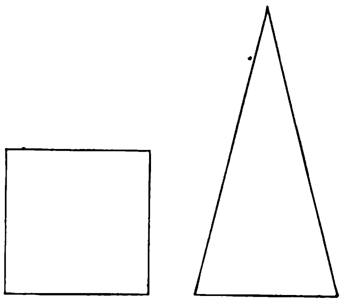


FIG. 153.

But what right have we to assert that the square and the triangle represented in Fig. 153 have exactly the same size? This question suggests the more general question: How can we compare as regards size two figures that differ in shape?

To answer this question for plane figures is the object of this chapter.

In other words, we are now to learn how plane figures are **measured.** In one respect the method resembles that employed for measuring lines. We first choose a suitable *unit*, and then proceed to find the number of times this unit is contained in the surface to be measured.

The number of units of surface thus found is called the **area** of the surface.

82. Units of Surface. The units of surface generally used are squares whose sides are equal to the units of length.

The English units of surface are:

The square inch (sq. in.).
The square foot (sq. ft.) = 144 sq. in.
The square yard (sq. yd.) = 9 sq. ft.
The square rod (sq. rd.) = $30\frac{1}{4}$ sq. yd.
The acre (A.) = 160 sq. rd.
The square mile (sq. mi.) = 640 A.

The most important metric units of surface are:

The square millimeter (qmm).
The square centimeter (qcm) = 100 qmm.
The square decimeter (qdm) = 100 qcm.
The square meter (qm) = 100 qdm.
The square kilometer (qkm) = 1,000,000 qm.
(Approximately, 1 qcm = $\frac{1}{6}$ sq. in.; 1 qm = $1\frac{1}{5}$ sq. yd.)

EXERCISES

1. One centimeter = 10 millimeters. Why does it follow that 1 square centimeter contains 100 square millimeters?

Fig. 154 should enable you to answer this question. Each side of the square *ABCD* is to be considered as representing 1 cm, and is divided into ten equal parts. Perpendiculars to the sides are drawn at all the points of division.

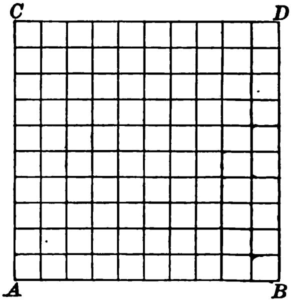


FIG. 154.

2. Explain by a diagram why a square foot contains 144 sq. in.

3. Why is a square yard equal to 9 sq. ft.?

4. Show that an acre contains 43,560 sq. ft.

83. Area of a Square. Let one side of a square (Fig. 155) be 4 in. long. Divide two adjacent sides into inches, and erect perpendiculars at all the points of division. The square is thus divided into 4×4 or 16 smaller squares, each equal to 1 sq. in. Therefore, the area of the given square is equal to 16 sq. in.

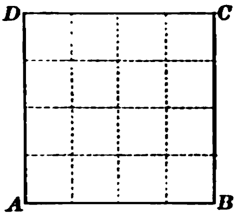


FIG. 155.

Suppose that one side of the square is 4.25 in. Proceed as before, only suppose each side divided into 425 equal parts, each $\frac{1}{100}$ of an inch long. Evidently we shall obtain 425×425 or 180,625 smaller squares. But it will take 100×100 or 10,000 of these little squares to make 1 sq. in. Therefore, the area of the square *in square inches* will be $\frac{180625}{10000}$ or 18.0625. This result is obtained by multiplying the number 4.25 by itself.

In general the area of a square is found by multiplying the number of units of length in one side of the square by itself; in other words, by *squaring* the number of units of length in one side.

This truth, for the sake of brevity, may be stated:

Area of a square = square of one side.

EXERCISES

Find the area of a square if:

1. One side = 9 in.
2. One side = 14 ft.
3. One side = $2\frac{1}{8}$ in.
4. One side = 15 cm.
5. One side = 25 m.
6. One side = 6.42 cm.
7. What will it cost at 20 cents a square foot to paint a square floor, the side of which is $18\frac{1}{2}$ ft. long?
8. Find the area in acres and in square feet of a square field, each side of which is 178 yd. long.

84. Area of a Rectangle. Let $ABCD$ (Fig. 156) be a rectangle in which $AB = 7$ in. and $AD = 4$ in. Then by the same method as in the case of a square it can be shown that the area of the rectangle is equal to 4×7, or 28, sq. in.

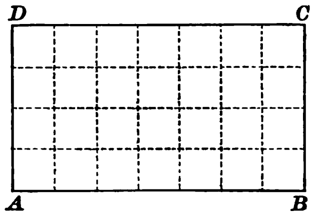


FIG. 156.

In every case the area of a rectangle is found by multiplying the number of units of length in one side by the number of units of length in the adjacent side. The two adjacent sides of a rectangle are usually called the *length* and the *breadth*.

We may express the above rule by the formula,

$$\textit{Area of a rectangle} = \textit{length} \times \textit{breadth}.$$

EXERCISES

Find the area of a rectangle, having given:

1. Length 12 in., breadth 9 in.
2. Length $15\frac{1}{2}$ in., breadth 6 in.
3. Length $3\frac{1}{2}$ in., breadth $2\frac{1}{4}$ in.
4. Length 200 ft., breadth 60 ft.
5. Length 200 ft., breadth 10 yd.
6. Length 16 cm, breadth 12 cm.
7. Length 40 cm, breadth 2 dm.

Find the breadth of a rectangle, having given:

8. Area 100 sq. in., length 20 in.
9. Area 288 sq. in., length 24 in.
10. Area $22\frac{1}{2}$ sq. in., length 9 in.
11. Area 180 sq. ft., length 6 yd.
12. Determine as accurately as you can the number of square inches of surface on the cover of this book.
13. Find the area of a football field, if the length is 110 yd. and the width is $53\frac{1}{3}$ yd.

14. How many bricks 9 in. by 4 in. are required to cover a floor 34 ft. long and 17 ft. wide?

NOTE. The expression "9 in. by 4 in." means "9 in. long and 4 in. wide." Therefore, the surface covered by each brick is 36 sq. in.

15. How many square feet of plank are required for a floor 20 ft. long and 14 ft. wide? How many boards, each 14 ft. long and 6 in. wide, are needed? How many yards of carpet 28 in. wide are required for covering the floor?

16. A lawn measuring 144 yd. by 98 yd. is to be covered with turf. Each turf is cut 18 in. long and 15 in. wide. The price to be paid is 75 cents a dozen laid down. Find the cost of turfing the lawn.

17. What length must be taken on a plank 18 in. wide to obtain just 4 sq. ft. of plank?

18. What must be the width of a field half a mile long to contain just 100 acres?

19. How is the area of a rectangle changed by doubling the length? By doubling the breadth? By doubling both the length and the breadth?

20. From a rectangular lot 528 ft. by 240 ft. there are sold four square lots, each having a side 60 ft. long. How much land remains?

21. A square and a rectangle have equal perimeters, 60 m. The rectangle is twice as long as it is wide. Which figure has the greater area, and what is the difference between the areas?

22. The sides of a rectangle are 4 in. and 9 in. Find the side of an equivalent square.

23. The sides of a rectangle are 4 in. and 10 in. Find the side of an equivalent square, correct to two decimal places.

24. A man has a rectangular field 400 ft. by 240 ft. If he makes streets 60 ft. wide directly through the field parallel to the sides, so as to divide the field into four equal rectangles, and divides each rectangle into two equal house lots, how much land will there be in each house lot? Draw a plan of the field divided as described, taking 1 cm to represent 25 ft. How many *square feet* does 1 qcm represent?

25. Find how many square feet of boarding will be required to make a fence 40 yd. long and 4 boards high, if the boards average 10 in. wide.

85. Area of a Parallelogram. Take any parallelogram, *ABCD* (Fig. 157), draw *AF* and *BE* perpendicular to *AB*, and extend *CD* to *F*. In this way we form a rectangle *ABEF*, having the same base and the same altitude as the parallelogram.

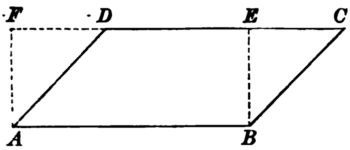


FIG. 157.

The triangles *AFD*, *BEC* are right triangles having equal legs *AF* and *BE*, and equal hypotenuses *AD* and *BC*. Therefore, the triangles are equal (p. 63). If the triangle *AFD* is taken from the whole figure *ABCF*, there is left the parallelogram; if the equal triangle *BEC* is taken from the whole figure, there is left the rectangle. Hence, the parallelogram and the rectangle are equivalent figures. The area of the rectangle is equal to $AB \times BE$ (p. 96). Therefore, the area of the parallelogram is equal to $AB \times BE$. In general,

Area of a parallelogram = base × altitude.

EXERCISES

Find the area of a parallelogram, having given:

1. Base 16 in., altitude 5 in.
2. Base $2\frac{1}{2}$ in., altitude 14 in.
3. Base 8.2 cm, altitude 6.7 cm.

4. Name in Fig. 158 three parallelograms and one rectangle. Compare their bases. Compare their altitudes. Compare their areas. What is true of parallelograms that have equal bases and equal altitudes?

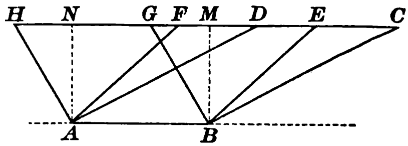


FIG. 158.

5. The bases of two parallelograms are equal. The altitude of one is four times that of the other. Compare their areas.

86. Area of a Triangle. A diagonal divides a parallelogram into two equal triangles (p. 63). Hence, any triangle ABC (Fig. 159) may be regarded as half of a parallelogram $ABCE$ having the same base AB and the same altitude as the triangle.

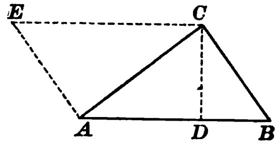


FIG. 159.

The area of a parallelogram = base × altitude (p. 98). Hence,

$$\textit{Area of a triangle} = \tfrac{1}{2}\ \textit{base} \times \textit{altitude}.$$

EXERCISES

Find the area of a triangle, having given:

1. Base 11 in., altitude 10 in.
2. Base $2\frac{1}{2}$ in., altitude 1 in.
3. Base 20 ft. 6 in., altitude 10 ft. 4 in.
4. Base 10.4 cm, altitude 8.6 cm.
5. Examine the triangles in Fig. 160. Have they the same base? Have they equal altitudes? Have they equal areas? What is true of triangles that have equal bases and equal altitudes?

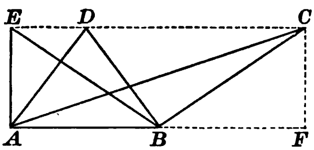


FIG. 160.

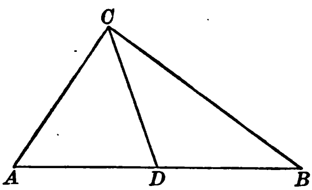


FIG. 161.

6. In a triangle ABC (Fig. 161), a straight line is drawn from the vertex C to the middle point D of the base AB. Compare the areas of the triangles ADC and BDC.
7. How can a triangle be divided into four equal triangles?
8. Can you explain how a triangle may be divided into two triangles such that one has double the area of the other?

87. Area of a Rhombus. The area of a rhombus may be found by multiplying its base by its altitude just as in the case of any other parallelogram. But the diagonals of a rhombus bisect each other at right angles (p. 63); and this fact gives us another way of finding the area of a rhombus.

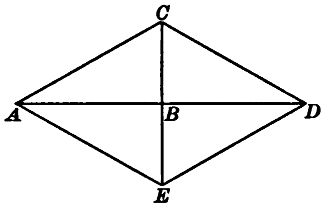


FIG. 162.

Let $ACDE$ (Fig. 162) be a rhombus. Draw the diagonals AD, CE. AD divides the rhombus into two equal isosceles triangles, ACD and AED, each having as base AD and as altitude half of CE. Therefore, the area of each triangle $= \frac{1}{2} AD \times \frac{1}{2} CE = \frac{1}{4} AD \times CE$. Hence, the area of the rhombus $= 2 \times \frac{1}{4} AD \times CE = \frac{1}{2} AD \times CE$. Hence,

Area of a rhombus = half the product of the diagonals.

EXERCISES

Find the area of a rhombus, having given:

1. The diagonals 6 in. and 8 in.
2. The diagonals $9\frac{3}{8}$ in. and $6\frac{1}{4}$ in.
3. The diagonals 330 ft. and 132 ft.
4. It is desired to lay out a park in the shape of a rhombus so that it shall contain 10 acres, and the longer diagonal shall be 2420 ft. long. Find the shorter diagonal, and draw a plan of the park, taking 1 cm to represent 200 ft.
5. Make a square by pinning together at the ends four narrow strips of cardboard. Then turn the sides about the pins so as to make the angles oblique. What kind of a figure is thus formed? If a square and a rhombus have equal sides, which has the greater area?
6. The legs of a right triangle are 6 in. and 8 in. Find the area.
7. Show that the area of a right triangle is equal to half the product of the legs.

88. Area of a Trapezoid. A trapezoid $ABCD$ (Fig. 163) is divided into two triangles ABD, BDC by drawing the diagonal BD. The base of the triangle ABD is AB, and the altitude is DE. The base of the triangle BDC is DC, and the altitude is BF, which is equal to DE.

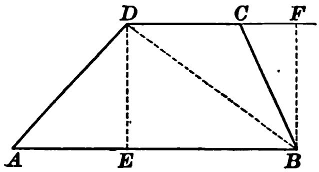


FIG. 163.

Area of triangle $ABD = \frac{1}{2} AB \times DE$.
Area of triangle $BDC = \frac{1}{2} DC \times BF = \frac{1}{2} DC \times DE$.

Adding, we have area of trapezoid $= \frac{1}{2} (AB + DC) \times DE$, a result which we may state as follows:

Area of a trapezoid $= \frac{1}{2}$ *(sum of bases)* $\times$ *altitude.*

EXERCISES

Find the area of a trapezoid, having given:

1. Bases 10 in. and 8 in., altitude 6 in.
2. Bases 20 cm and 30 cm, altitude 5 cm.
3. Bases 37 ft. and 25 ft., altitude 19 ft.
4. Find in square miles the area of the trapezoid $ABCD$ (Fig. 163), if $AB = \frac{1}{2}$ mile, $CD = \frac{1}{4}$ mile, $DE = \frac{1}{8}$ mile.
5. The four lateral faces of the pedestal of a statue are equal trapezoids (Fig. 164). The bases of each trapezoid are 20 ft. and 15 ft. in length. The altitude of each trapezoid is equal to 8 ft. What is the entire lateral surface?

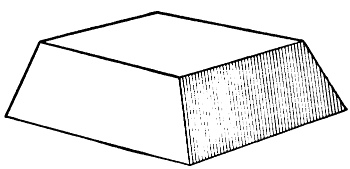


FIG. 164.

6. A playground has the shape of an isosceles trapezoid. The bases are 600 ft. and 200 ft. in length, and the larger base makes angles of 60° with the legs. Construct a plan of the playground, taking 1 cm to represent 50 ft., and find the area of the playground in square feet.

89. Area of a Regular Polygon. The straight lines drawn from the centre of a regular polygon (Fig. 165) to the vertices divide the polygon into as many equal isosceles triangles as the polygon has sides. The base of each triangle is a side of the polygon, and its altitude is equal to the radius of the inscribed circle.

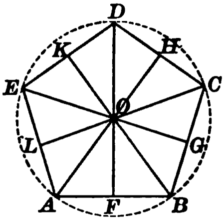


Fig. 165.

If we find the area of one of these triangles and multiply it by the number of sides of the polygon, we evidently obtain the area of the polygon.

Approximate values of the radius of the inscribed circle for several regular polygons are given in the following table:

No. of sides.	Radius.	No. of sides.	Radius.
3	0.288 × one side	7	1.038 × one side
4	0.500 × one side	8	1.207 × one side
5	0.688 × one side	10	1.539 × one side
6	0.866 × one side	12	1.866 × one side

EXERCISES

Find the area of:

1. An equilateral triangle, if one side is 5 in.
2. A regular pentagon, if one side is 6 in.
3. A regular octagon, if one side is 15 ft.
4. A regular decagon, if one side is 19 ft.
5. A park has the shape of a regular hexagon. Each side is 1000 ft. long. What is the assessed valuation at 8 cents per square foot?
6. Find the total area covered by the shaded triangles in Fig. 142, p. 88, if a side of each triangle is 2 ft. long.
7. Find the total area covered by the shaded squares in Fig. 143, p. 88, if a side of each square is 2 ft. long.
8. One side of a regular hexagon is 1 ft. long. Find the side of a square equivalent to the hexagon.

90. Area of any Polygon. The area of any polygon may be found by dividing the polygon into simpler figures (triangles, rectangles, trapezoids, etc.), as illustrated by the following exercises, then computing the area of each part, and adding the results.

EXERCISES

1. Find the area of the polygon $ABCD$ (Fig. 166), having given: AC 400 yd.; BE 120 yd.; DF 80 yd.

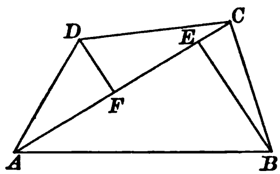


FIG. 166.

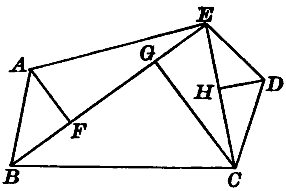


FIG. 167.

2. Find the area of the polygon $ABCDE$ (Fig. 167), having given: BE 108 yd.; EC 96 yd.; AF 49 yd.; HD 35 yd.; CG 67 yd.

3. Find the area of the polygon $ABCDE$ (Fig. 168), having given: AC 550 yd.; BG 160 yd.; AF 160 yd.; AH 500 yd.; EF 160 yd.; DH 90 yd.

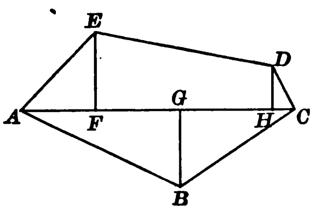


FIG. 168.

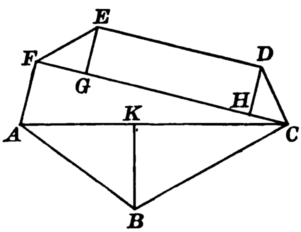


FIG. 169.

4. Find the area of the polygon $ABCDEF$ (Fig. 169), having given: AC 340 yd.; BK 115 yd.; FG 50 yd.; EG 54 yd.; FH 283 yd.; DH 60 yd.; AF 80 yd.; CF 330 yd.; the angle $AFC = 90°$.

What other mode of division might be used for this polygon?

91. Theorem of Pythagoras. The following very useful truth was discovered more than 2000 years ago by the Greek philosopher Pythagoras :

The square of the hypotenuse of a right triangle is equal to the sum of the squares of the two legs.

A special case of the theorem is illustrated in Fig. 170. The sides of the right triangle contain 3, 4, and 5 units of length, respectively. The squares constructed upon the sides contain 9, 16, and 25 units of surface, respectively. Since $25 = 9 + 16$, the theorem evidently holds true in this case.

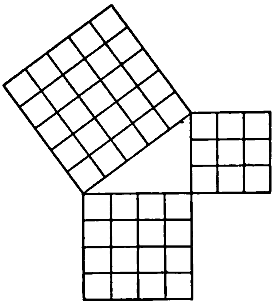


FIG. 170.

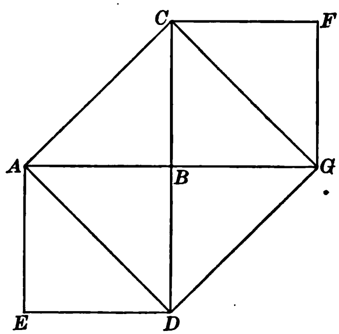


FIG. 171.

Consider the case of a right triangle ABC having equal legs AB and BC (Fig. 171). The acute angles at A and C are each 45°. If we construct upon the legs the squares $ABDE$ and $BCFG$, and draw DG, AD, and CG, we obtain a figure $ACFGDE$ divided into six equal right triangles, each equal to the triangle ABC. The square constructed upon AC contains four of these triangles, and each of the squares constructed upon the legs contains two of these triangles. Therefore, the theorem holds true in this case.

The theorem can be proved true for any right triangle.

EXERCISES

1. If the legs of a right triangle are 6 cm and 8 cm long, find the hypotenuse.

2. If the legs of a right triangle are 8 cm and 15 cm long, find the hypotenuse.

3. The hypotenuse of a right triangle is 20 cm long, and one leg is 16 cm long. What is the length of the other leg?

4. Find the length of one leg of a right triangle if the lengths of the other leg and the hypotenuse are 5 cm and 13 cm.

5. A ladder 34 ft. long just reaches a window when placed with its foot 16 ft. from the side of the house. How high is the window above the ground?

6. How long must a ladder be to reach the top of a wall 24 ft. high, if the foot of the ladder is 10 ft. from the bottom of the wall?

7. The longest side of a meadow in the shape of a right triangle cannot be measured directly on account of the swampy nature of the ground. What is its length, if the other sides are 960 ft. and 720 ft. long?

8. Two vessels start at the same time from the same place. One sails due north at the rate of 6 miles an hour, and the other sails due east at the rate of 8 miles an hour. How far apart are they at the end of 6 hours?

9. Find the area of a right triangle if the length of the hypotenuse is 25 in. and the length of one leg is 15 in.

10. Find to two decimal places the hypotenuse of a right triangle, each leg of which is 20 cm long.

11. Find to two decimal places the legs of an isosceles right triangle, if the hypotenuse is 20 cm long.

12. Find the area of an isosceles triangle if one leg is 10 ft. long and the base is 12 ft. long.

13. The side of an equilateral triangle is 20 ft. Find the altitude and the area of the triangle.

14. The side of a square is 20 ft. Find the length of the diagonal.

15. The diagonal of a square is 20 ft. Find the length of one side.

16. How far apart are the opposite corners of a floor that is 24 ft. long and 18 ft. wide?

92. Area of a Circle. Consider a regular polygon circumscribed about a circle (Fig. 172), and divided into equal isosceles triangles.

The area of each triangle is equal to the product of half of one side of the polygon and the radius of the circle. It follows that the area of the polygon is equal to the product of *half its perimeter and the radius of the circle.* Now the area of the polygon is greater than the area of the circle. Also, the more sides there are to the polygon the more nearly its area approaches the area of the circle. Starting from these obvious facts, it can be proved that *the area of the circle is equal to the product of half the circumference and the radius.*

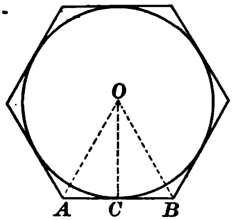


FIG. 172.

Approximately (p. 91) the circumference $= \frac{22}{7} \times$ diameter. Hence, *half* the circumference $= \frac{22}{7} \times$ radius.

Therefore, *the area of a circle* $= \frac{22}{7} \times$ *square of radius*, or if instead of $\frac{22}{7}$, we take the decimal value 3.1416,

Area of a circle $= 3.1416 \times$ *square of radius.*

REVIEW EXERCISES

Find the area of a circle, having given:

1. Radius 7 in.
2. Radius 14 ft.
3. Radius 20 cm.
4. Radius 1 m.
5. Diameter 49 ft.
6. Diameter $3\frac{1}{2}$ ft.
7. Diameter 16 cm.
8. Diameter 100 in.
9. The diameter of a park in the shape of a circle is half a mile. Find the number of acres in the park.
10. A horse is tied to a stake on a grass plot by a rope 42 ft. long. Over how many square feet of grass can he feed?
11. The radii of two concentric circles are 14 ft. and 21 ft. Find the area of the ring bounded by the circumferences.

12. A circular fort 56 ft. in diameter is surrounded by a moat 14 ft. wide, full of water. Find the area of the moat.

13. There are two circular lots of land. The radius of one lot is just half that of the other lot. If the first lot is worth $100, what is the second lot worth?

14. How many circles, each having a radius of 3 in., will be required to obtain an area as large as that of a single circle whose radius is 12 in.?

15. Find the area of the largest circle that can be cut from a square piece of wood, each side of which is 2 ft. 4 in. long.

16. The circumference of a circle is equal to the perimeter of a square. Which figure has the greater area? Show by an example that your answer is correct.

17. The area of a circle is 100 sq. ft. Find the radius.

Solution. $100 = \frac{22}{7} \times (\text{radius})^2;$

$$(\text{radius})^2 = \frac{7 \times 100}{22} \text{ ft.} = 31.8181 \text{ ft.};$$

$$\text{radius} = \sqrt{31.8181} \text{ ft.} = 5.64 \text{ ft., nearly.}$$

18. What must be the diameter of a circular field that the area may be 6 acres?

19. Find the radius of a circle, if the area is 44 sq. in.

20. Find the radius of a circle equivalent to a square, each side of which is 11 ft. long.

21. The radius of a circle is 4 in. Find the radius of a circle whose area is twice as large as the area of the given circle; also the radius of a circle whose area is four times as large as the area of the given circle.

22. Find the side of a square equivalent to a circle, the diameter of which is 84 ft. long.

23. Find the diameter of a circle equivalent to a square, one side of which is 7 ft. long.

24. Find the number of acres of land enclosed by a circular race track 1 mile long.

25. The diameter of a circular reservoir is 64 yd. Around the reservoir a walk 1 yd. wide is constructed, and outside the walk a fence is built, find the length of the fence and the area of the walk.

26. Find the area of a circular sector, if the radius of the circle is 14 in., and the angle of the sector is 45°.

Since the circumference contains 360°, it is clear that the sector of 45° is $\frac{45}{360}$, or $\frac{1}{8}$ of the area of the entire circle (Fig. 173).

Find the areas of the following sectors:

27. Radius 7 in., angle 30°.

28. Radius 35 in., angle 120°.

29. Diameter 28 ft., angle 60°.

30. The radius of a circle is 42 in. Find the area of a sector whose arc is 22 in. long.

The area of the whole circle is half the circumference × the radius, and the area of any sector is half its arc × the radius.

31. Find the area of a sector whose radius is 14 in. long, and whose arc is 11 in. long.

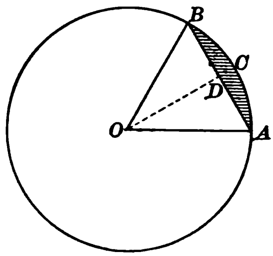


FIG. 173.

What is the value of the angle of this sector?

32. What part of the entire circle is a sector, the chord of whose arc is equal in length to the radius of the circle? What is the value of the angle of this sector?

33. Find the area of the *segment* ACB (Fig. 173), if the radius of the circle is 7 in., and the angle AOB is 60°.

Evidently segment ACB = sector AOB – triangle AOB.

In this case the triangle AOB is equilateral, and each side is equal to the radius of the circle. The altitude OD of the triangle bisects the chord AB, and its length can be found by applying the Theorem of Pythagoras (p. 104).

34. The radius of a circle is equal to 56 ft., and the central angle corresponding to a segment of the circle is 90°. Find the area of the segment.

35. The diameter of the cross-section of a wooden log is 3 ft. Find the area of the cross-section of the largest square beam that can be sawed from the log.

CHAPTER VI

SIMILAR FIGURES

93. Numerical Measures. A magnitude is measured by finding the number of times it contains another magnitude of the same kind taken as a unit of measure. The number of times the unit is contained in the magnitude is called the **numerical measure** of the magnitude with reference to the unit.

If MN (Fig. 174) is taken as the unit of length, and if AB, BC, etc., are each equal to MN, then the numerical measure of AB with reference to MN is 1, that of AC is 2, etc.

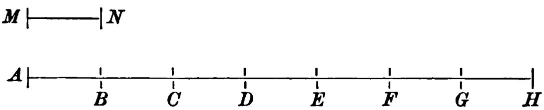


FIG. 174.

94. Ratio. The **ratio** of two quantities of the same kind is the quotient of the numerical measure of one quantity divided by the numerical measure of the other, when both quantities are expressed in terms of the same unit.

For example (Fig. 174), the ratio of AD to AG is $\frac{3}{6}$ or $\frac{1}{2}$, the ratio of AG to AD is $\frac{6}{3}$ or $\frac{2}{1}$, the ratio of AC to AD is $\frac{2}{3}$.

A ratio is often expressed by placing a colon between the two numbers or *terms* of the ratio. Thus, the ratio of AD to AG is written $3:6$, and read "as 3 is to 6," or "the ratio of 3 to 6."

Find the ratio of two straight lines AB, CD (Fig. 175).

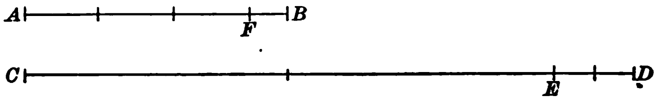


FIG. 175.

First Method. Measure each line to the nearest millimeter with a graduated ruler, and divide one result by the other.

Second Method. Apply by means of dividers AB to CD as many times as possible. In this case AB can be applied twice, and there is a remainder ED. Now apply this remainder ED to AB as many times as possible; in this case three times with the remainder FB. Apply this remainder FB to the remainder ED; in this case exactly twice. Then

$$ED = 2FB;$$

$$AB = 3ED + FB = 7FB;$$

$$CD = 2AB + ED = 14FB + 2FB = 16FB.$$

Therefore, $$\frac{AB}{CD} = \frac{7FB}{16FB} = \frac{7}{16},$$

or $AB : CD = 7 : 16$.

Here FB is a *common measure* for the lines AB and CD.

If two lines have no common measure, they are said to be *incommensurable*. The side and the diagonal of a square are incommensurable lines.

EXERCISE

1. What is the ratio (Fig. 174) of AB to AC? of AC to AB? of AC to AD? of AC to AG? of AD to AH? of AH to AF?

95. Proportion. Four quantities are said to be **proportional,** or to form a **proportion,** if the ratio of the first to the second is equal to the ratio of the third to the fourth.

The four quantities are called the **terms** of the proportion; the first and the fourth are called the **extremes**; the second and the third are called the **means.**

Thus, the numbers 2, 5, 4, 10 are proportional numbers; the proportion is written $2:5=4:10$, and is read "2 is to 5 as 4 is to 10"; 2 and 10 are the extremes, 5 and 4 the means.

The lines AB, CD (Fig. 176) have the ratio $3:5$; and the lines EF, GH (Fig. 177) have the same ratio. The four lines AB, CD, EF, GH form the proportion $AB : CD = EF : GH$.

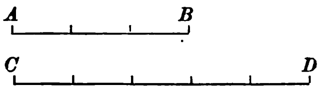


FIG. 176.

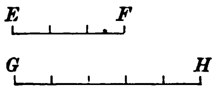


FIG. 177.

In every proportion the product of the extremes is equal to the product of the means.

Thus, in the proportion $2:5=4:10$, the product of the extremes is 20, and the product of the means is 20.

EXERCISES

1. Express in lowest terms the ratio of 6 in. to 24 in., 4 ft. to 6 in., 15° to 90°.

2. If the first three terms of a proportion are 36, 9, and 28, find the fourth term.

Let x denote the fourth term. Then $x = \dfrac{9 \times 28}{36} = 7$.

Find the missing term in the following proportions:

3. $6:12=20:x$.
4. $4:24=x:90$.
5. $1:x=9:36$.
6. $x:40=17:68$.
7. $2.5:7.5=0.5:x$.
8. $3\frac{1}{8}:28\frac{1}{8}=1:x$.

96. In certain cases it can be proved that the ratio of two areas is equal to the ratio of two lines. By the use of proportions of this kind the solution of numerical questions respecting areas is often simplified.

EXERCISES

1. Two triangles ADC, BDC (Fig. 178) have the same altitude CE. Their bases are 13 ft. long and 20 ft. long. Compare their areas. We know (p. 99) that

triangle $ADC = 13 \times \frac{1}{2} CE$,
and triangle $BDC = 20 \times \frac{1}{2} CE$.

Therefore, $\dfrac{\text{triangle } ADC}{\text{triangle } BDC} = \dfrac{13 \times \frac{1}{2} CE}{20 \times \frac{1}{2} CE} = \dfrac{13}{20}$.

That is,

triangle ADC : triangle $BDC = 13 : 20$.

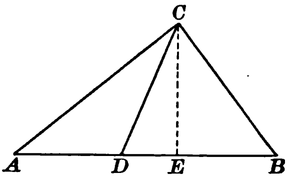


FIG. 178.

2. If the area of the triangle ADC (Fig. 178) is 208 sq. ft., find the area of the triangle BCD.

3. Compare (Fig. 178) the areas of the triangles ABC and ADC; also the areas of the triangles ABC and BDC.

4. The bases of two triangles that have equal altitudes are 5 ft. and 15 ft., respectively. If the smaller triangle contains 100 sq. ft., what is the area of the larger triangle?

5. What is always the ratio of the areas of two triangles that have equal altitudes but unequal bases?

6. A triangular field ABC contains just 8 acres. The owner wishes to sell 2 acres of it, and to run the division line straight from C to the proper point D in AB. Show how to locate the point D without making any measurements in the field.

7. Two triangles have equal bases. The altitude of one is 10 ft.; that of the other is 60 ft. Show by proceeding as in Ex. 1 that one triangle is six times as large as the other.

8. The altitudes of two triangles that have equal bases are 7 ft. and 63 ft. The area of the smaller triangle is 84 sq. ft. Find the area of the larger triangle, and find the common base.

9. What is always the ratio of the areas of two triangles that have equal bases but unequal altitudes?

10. Draw any triangle and then construct in the simplest way another triangle which shall be three times as large.

11. Two parallelograms have equal altitudes, but the base of one is ten times as long as that of the other. Compare their areas.

12. A triangle and a parallelogram have equal bases and equal altitudes. What is the ratio of their areas?

13. A triangle and a parallelogram have equal bases and equal areas. What is the ratio of their altitudes?

14. A triangle and a parallelogram have equal bases. The altitude of the triangle is four times that of the parallelogram. Compare their areas.

15. It is desired to run a division line AE (Fig. 179) straight across a rectangular field $ABCD$, so that the triangle ADE cut off shall be equal in area to $\frac{1}{8}$ of the entire field. How would you determine the proper position of the point E? What is the ratio of DE to DC? What is the ratio of DE to EC?

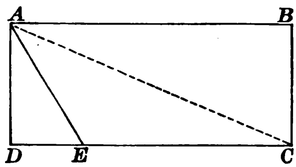


FIG. 179.

Locate the point E (Fig. 179) in order that:

16. Triangle ADE shall be one fourth the rectangle $ABCD$.

17. Triangle ADE : rectangle $ABCD = 1:12$.

18. Triangle ADE : rectangle $ABCD = 3:7$.

19. The radii of two circles are 1 ft. and 3 ft., respectively. Find the ratio of the circumferences and the ratio of the areas.

20. If the radius of a circle is doubled, what change takes place in the circumference? What change in the area?

21. The diameters of two circles are as 1 to 5. What is the ratio of their circumferences? What is the ratio of their areas?

22. The diameters of two circular flower beds are as 3 to 4. The area of the smaller bed is 90 sq. ft. What is the area of the other?

23. The radii of three circular water tanks are 6 ft., 9 ft., and 15 ft., respectively. If a mason charges \$5 for cementing the bottom of the smallest tank, what ought he to charge for cementing the bottom of each of the other tanks?

97. Draw an angle XAY (Fig. 180), lay off on one side any equal lengths AB, BC, CD, etc. Through the points B, C, D, etc., draw any parallel lines, cutting the side AY at the points E, F, G, etc. The more carefully you draw these parallels the more nearly you will find by using dividers that the lengths AE, EF, FG, etc., are also equal. In this way you will doubtless become convinced of the truth of the following theorem, although the proof is not given here.

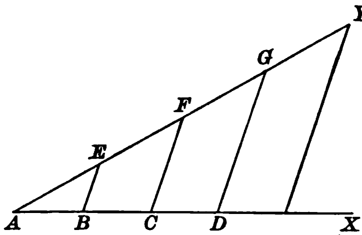


Fig. 180.

If parallel lines intercept equal lengths on one side of an angle, they also intercept equal lengths on the other side of the angle.

This theorem is one of the most useful truths of geometry. Upon it is based the division of a straight line into equal parts (p. 69), and it leads directly to the idea of proportional lines, and to the solution of various problems of great practical importance.

It is easy to find proportional lines in Fig. 180.

Since $AB = BC = CD$,
we have $AB : BD = 1 : 2$.
And since $AE = EF = FG$,
we have $AE : EG = 1 : 2$.
Therefore, $AB : BD = AE : EG$.
Again, $AB : AD = 1 : 3$, and $AE : AG = 1 : 3$.
Therefore, $AB : AD = AE : AG$.

Show by reasoning in a similar manner that

$$AC : CD = AF : FG.$$

98. ***A straight line drawn parallel to one side of a triangle, cutting the other two sides, divides those sides proportionally.***

This follows from the theorem on page 114.

For example, let DE (Fig. 181) be drawn parallel to the side BC of the triangle ABC.

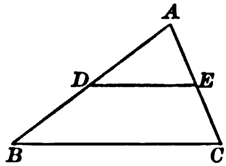


FIG. 181.

Then $$AD:DB = AE:EC,$$
$$AD:AB = AE:AC,$$
and $$AB:DB = AC:EC.$$

Upon this truth depends the solution of the following problems.

EXERCISES

1. Divide a given straight line AB (Fig. 182) into two parts that shall be to each other as 3 to 4.

Draw through A a straight line AC, making an acute angle with AB. Upon AC lay off $3 + 4$ or 7 equal parts ending at the point D. Draw DB. Draw through E the third division point a line EF, parallel to DB, meeting AB at F.

Then $AF:BF = 3:4$.

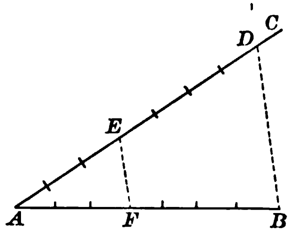


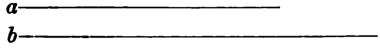
FIG. 182.

2. Divide a straight line into two parts that have the ratio $4:9$.

3. Produce a straight line AB to a point H, so that
$$AB:BH = 7:5.$$

4. Divide a straight line into three parts which shall be to each other as the numbers 2, 3, 5.

5. Draw a straight line AB and then divide it into two parts that shall have to each other the same ratio as the two lines a and b given below.



Hint. Draw a straight line making an acute angle with AB, and lay off upon it the lengths a and b.

6. Draw a straight line, and then divide it into two parts such that one part shall be two and one half times as long as the other.

7. Construct a straight line the length of which shall be the fourth proportional to the lengths m, n, p of the three unequal straight lines (Fig. 183).

First Method. Measure the lengths m, n, p as closely as possible; suppose them to be 3.2 cm, 4.1 cm, and 4.8 cm.

Then $\qquad 3.2 : 4.1 = 4.8 : x.$

Hence, $\qquad 3.2\,x = 4.1 \times 4.8,$

and $\qquad x = 6.15$ cm.

Then draw a straight line 6.15 cm long.

This may be called the *arithmetical* solution of the problem.

Second Method. Draw any acute angle xAy (Fig. 183).

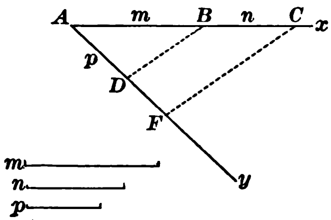
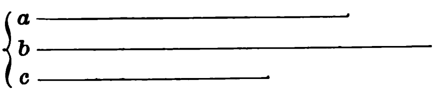


FIG. 183.

From A on Ax take AB equal to m, and BC equal to n. From A on Ay take AD equal to p. Draw DB. Draw CF parallel to DB. The length DF is the fourth proportional to the lengths m, n, p; or $m : n = p : DF$.

This is the *geometrical* solution of the problem.

Find by both the above methods the fourth proportional to the following lines. Which method do you prefer? Give your reason.

8.


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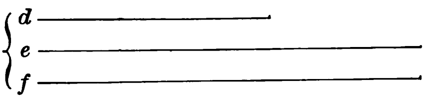

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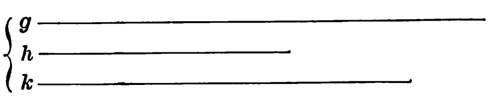

99. Similar Polygons. Similar polygons are polygons that have the same shape.

Let us first consider similar triangles. The best way to see clearly their two most important properties is illustrated by Fig. 184. Draw any triangle ABC. Divide the side AB into equal parts; say five parts. Through the points of division D, E, F, G, draw parallels to the side BC, and also parallels to the side AC.

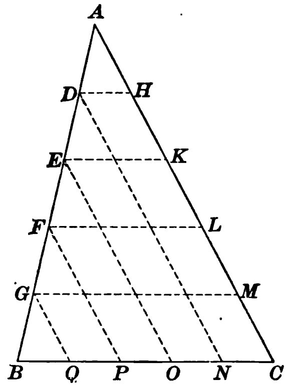

FIG. 184.

Now compare one of the triangles thus formed, AEK, with the triangle ABC. It has the same shape as ABC, and is, therefore, similar to ABC; and it would remain similar to ABC if it were removed to any other position.

Compare the angles of the two triangles. The angle A is common. The angles AEK and ABC are equal because they are exterior-interior angles of parallel lines (p. 54). The angles AKE and ACB are equal for the same reason. Thus, *the angles of the two triangles, taken pair by pair in the same order, are equal;* that is, *the two triangles are equiangular with respect to each other.*

Examine next the sides, and call the sides opposite the equal angles **homologous sides.** There are three pairs of homologous sides: AE and AB, AK and AC, EK and BC.

Now $\quad AE:AB = 2:5;\ AK:AC = 2:5;$

and $\quad EK:BC = 2:5.$

Therefore, $AE:AB = AK:AC = EK:BC.$

That is, *the homologous sides are proportional.*

Thus, it appears that two similar triangles have the following properties:

1. *They are equiangular with respect to each other.*
2. *Their homologous sides are proportional.*

It can be proved that if two triangles have either one of these properties they have the other also and are similar.

If, for instance, we have two triangles which are equiangular with respect to each other, they are similar; and if we have two triangles which have their homologous sides proportional, they are similar.

EXERCISES

Compare the following pairs of triangles in Fig. 184, showing that they have properties 1 and 2 above stated, and find the common ratio of their homologous sides:

1. Triangles ADH and ABC.
2. Triangles ADH and AGM.
3. Triangles AEK and AFL.
4. The sides of a triangle are 4 cm, 7 cm, and 9 cm in length. In a similar triangle the side homologous to the side 4 cm long is 12 cm long. Find the length of the other two sides.
5. The legs of a right triangle are 6 cm and 8 cm long. A similar right triangle has its shorter leg 30 cm long. What is the length of its other leg and of its hypotenuse? Find the ratio of the perimeters of the two triangles.
6. Draw any triangle and then construct a similar triangle, each side of which shall be one fourth as long as the homologous side of the first triangle. What is the ratio of the perimeters?
7. Draw any triangle and then construct a similar triangle each side of which shall be three fourths as long as the homologous side of the first triangle. What is the ratio of the perimeters?
8. Draw a triangle ABC, and a straight line DE. Then, taking DE as homologous to AB, construct a triangle DEF similar to the triangle ABC.

9. Construct a straight line, the length of which shall be the *mean proportional* between the lengths of two given straight lines, a and b (Fig. 185); in other words, find a length c, such that

$$a : c = c : b, \text{ or } c^2 = ab.$$

Solution. Draw a straight line and lay off upon it AB equal to a, and BC equal to b. Upon AC as a diameter describe a semicircle. At the point B erect a perpendicular meeting the semicircumference at D.

BD is the length required.

Proof. The triangles ABD, CBD have right angles at B, and the angles BAD and BDC are equal, because each is equal to $90° - ADB$. Therefore, the third angles BDA and BCD are equal. Hence, the triangles are similar.

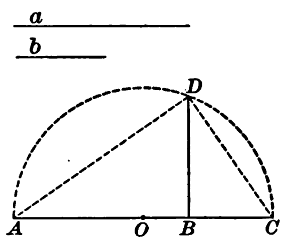

FIG. 185.

$$\text{Therefore, } AB : BD = BD : BC, \text{ or } \overline{BD}^2 = AB \times BC.$$

That is, if BD is denoted by c, $c^2 = a \times b$.

In general, the *mean proportional* between two numbers is a number whose square is equal to their product. For example, the mean proportional between 4 and 9 is 6.

10. Find the mean proportional between 9 and 16.

11. Find the mean proportional between 16 and 25.

12. Draw a rectangle, and then construct a square equivalent to the rectangle.

Hint. Find the mean proportional between the adjacent sides of the rectangle.

13. Draw a parallelogram, and then construct a square equivalent to it.

Hint. Find the mean proportional between the base and altitude of the parallelogram.

14. Draw a triangle, and then construct a square equivalent to it.

Hint. Find the mean proportional between the altitude and half the base.

15. Draw a trapezoid, and then construct a square equivalent to it.

Hint. Find the mean proportional between the altitude and half the sum of the bases.

100. Any two similar polygons (Figs. 186, 187) satisfy the two following conditions:

1. *For every angle in one of the polygons there is an equal angle in the other.*

2. *The homologous sides of the polygons are proportional.*

The equal angles, pair by pair, are called corresponding or homologous angles; and the sides which lie between the vertices of homologous angles, pair by pair, are called corresponding or homologous sides. Thus the angles EAB and $E'A'B'$, B and B', BCD and $B'C'D'$ are homologous angles, and the sides AB and $A'B'$, BC and $B'C'$, CD and $C'D'$ are homologous sides.

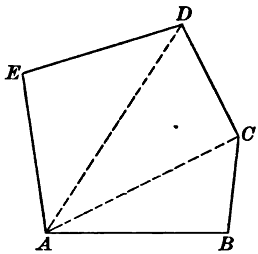

FIG. 186.

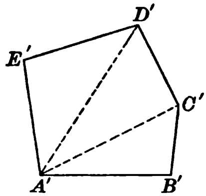

FIG. 187.

Two polygons may be equiangular with respect to each other, and yet not be similar; a square and a rectangle, for example, are not similar. Likewise, two polygons may have their sides, taken in the same order, proportional, and yet not be similar; a square and a rhombus, for example, are not similar. Before we can say that two polygons are similar, we must be sure that they satisfy each of the two conditions mentioned above.

Triangles are unlike other polygons in this respect. If two triangles satisfy either of the two conditions, they satisfy the other condition, and are similar.

101. It can be proved that

Two polygons composed of the same number of triangles, similar pair by pair and similarly placed, are similar.

EXERCISES

1. Construct a polygon similar to a given polygon $ABCDE$ (Fig. 186), having given the side $A'B'$ corresponding to AB.

Solution. Draw the diagonals AC and AD.

Construct the triangle $A'B'C'$ similar to the triangle ABC. Then construct the triangle $A'C'D'$ similar to the triangle ACD, and the triangle $A'D'E'$ similar to the triangle ADE. The polygon $A'B'C'D'E'$ is the polygon required.

2. Construct a quadrilateral similar to a given quadrilateral $ABCD$ (Fig. 188), having each of its sides three fourths as long as the corresponding side of the given quadrilateral.

Draw the diagonal AC.

On AB lay off a length AE equal to three fourths of AB.

Draw EF parallel to BC, meeting AC at F. Draw FG parallel to CD, meeting AD at G. The quadrilateral $AEFG$ is the quadrilateral required.

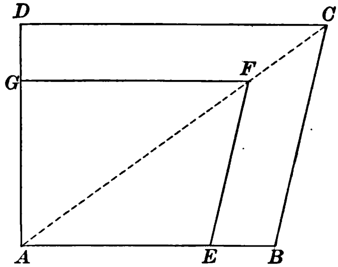

FIG. 188.

3. Construct a square, and then construct a square each side of which shall be two thirds as long as a side of the first square.

4. Construct two similar reetangles, one with sides 4 cm and 7 cm long, the other with sides twice as long.

5. Construct two similar rhombuses, having for angles 45° and 135°; and for homologous sides 6 cm and 12 cm.

6. Construct three similar parallelograms, having for angles 60° and 120°; and their homologous sides proportional to the numbers 1, 2, and 3.

7. Construct two similar isosceles trapezoids so that their corresponding sides shall have the ratio 1 to 2.

What is the ratio of their perimeters?

102. Areas of Similar Figures. If the side of one square is double the side of another square, the ratio of their areas is 1 : 4; that is, as the square of 1 is to the square of 2.

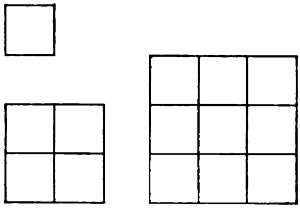

FIG. 189.

If the ratio of the sides of two squares is 2 : 3, the ratio of their areas is 4 : 9 (Fig. 189).

That is, the areas of two *squares* are to each other in the same ratio as the squares of the lengths of their sides.

Consider the similar triangles *ABC*, *ADE* (Fig. 190). Their homologous sides are to each other as 1 : 2. But their areas are as 1 : 4; this is evident if we divide the triangle *ADE* into four equal triangles as shown in the figure.

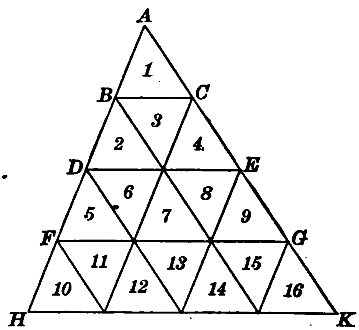

FIG. 190.

Consider the similar triangles *ABC*, *AFG*. Their homologous sides are to each other as 1 : 3; their areas are as 1 : 9.

Consider the similar triangles *ADE*, *AFG*.

Their homologous sides are to each other as 2 : 3.

Their areas are to each other as 4 : 9.

Again, the homologous sides of the similar triangles *ADE*, *AHK* are to each other in the ratio of 2 : 4, and their areas are in the ratio of 4 : 16. It follows, therefore, that

The areas of *two similar triangles* are to each other as the squares of any two homologous sides. In general,

The areas of two similar polygons are to each other as the squares of two homologous sides.

EXERCISES

1. The homologous sides of two similar triangles are 2 ft. and 100 ft. The area of the smaller triangle is 3 sq. ft. What is the area of the larger triangle?

2. One side of a triangle is 5 ft. long. Find the homologous side of a similar triangle twenty-five times as large.

3. The sides of a triangle are: $AB = 400$ ft., $AC = 500$ ft., $BC = 600$ ft. The triangle is to be divided into two equivalent parts by a straight line DE parallel to BC.

Find the lengths of AD and AE.

Hint. The triangles ADE and ABC (Fig. 191) are similar, and their areas are as 1 to 2.

Therefore, $\overline{AD}^2 : \overline{AB}^2 = 1 : 2.$

Hence, $AD = \dfrac{AB}{\sqrt{2}} = \dfrac{AB\sqrt{2}}{2}$

$= AB \times 0.707.$

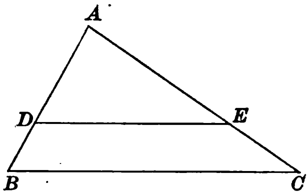

FIG. 191.

4. Explain how to draw a line parallel to one side of a triangle so as to cut off a smaller triangle equal in area to one fourth of the given triangle.

103. Any two equilateral triangles have the same shape and are similar figures. The same is true of any two squares. It can be proved that

Any two regular polygons having the same number of sides are similar figures.

EXERCISES

1. The sides of two equilateral triangles are 2 ft. and 12 ft. in length. What is the ratio of their areas?

2. What is the easiest way to construct a regular pentagon one fourth as large as a given regular pentagon?

3. Two parks have the shape of regular hexagons. If their sides are 1000 ft. and 8000 ft., respectively, what is the ratio of their areas? What is the ratio of their perimeters?

104. Drawing to Scale. In drawing to scale, the properties of similar figures are usefully applied. A plan of a field, for example, is a figure exactly resembling the field in shape, but much smaller in size. It is a reduced copy of the field. It is made by reproducing the angles of the field without change, but reducing all lines in a fixed ratio.

Thus, if a line on the ground 160 m long is represented by a line 0.16 m or 16 cm long, a line on the ground 800 ft. long must be represented by a line 0.8 ft. or 9.6 in. long, and so on.

A common mode of reducing lengths in a given ratio is by means of a **plotting scale**. A portion of a plotting scale, divided decimally, is shown in Fig. 192.

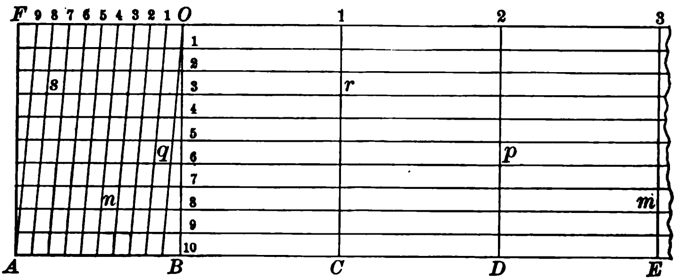

Fig. 192.

This scale is constructed as follows. Draw a straight line and take on it equal parts *AB*, *BC*, *CD*, etc. Let each part represent 100 units of length (feet, meters, etc.). Erect perpendiculars at the points *A*, *B*, *C*, etc. Draw ten straight lines parallel to *AD*, and equidistant from one another, as shown in the figure. Divide *AB* and *FO* each into ten equal parts. Join the points of division of *OF* to those of *AB* by lines drawn obliquely as shown.

Number the points of division between O and F from 1 to 10; also number the points of intersection of OB with the parallels in the same way.

Then from the properties of similar triangles it follows that the distance from the point 1 on OB, measured on the parallel through 1, to the first oblique line is equal to 1 unit of length; the distance from 1 to the second oblique line is equal to 11 units of length; the distance from 2 to the first oblique line is equal to 2 units of length; the distance from 2 to the second oblique line is equal to 12 units of length; and so on.

The point O is the zero of the scale; hundreds are read off to the right, tens to the left, and units on the vertical line OB.

From this explanation it is easy to see that a length of 348 m measured on the ground is represented on the scale by the line mn; that the length pq corresponds to a distance on the ground of 216 m; the length rs, to 183 m, etc.

EXERCISES

1. If the scale of reduction is 4 in. to 1 mile, what lengths on paper will represent an actual length of 4 miles? $2\frac{1}{2}$ miles? $\frac{1}{2}$ mile? 1320 ft.? 660 ft.? 880 yd.? 2000 ft.?

2. If the scale of reduction of a map is known to be 2 in. to the mile, and the distance between two cities on the map is $6\frac{1}{2}$ in., what is the actual distance between the cities?

3. Construct with the help of the scale in Fig. 192 a triangle the lengths of whose sides are 137 m, 160 m, and 225 m.

4. A certain field $ABCDE$ has the shape of a pentagon. Its angles are: A 90°; B 100°; C 120°; D 140°; E 90°. The lengths of three sides are: AB 417 m; BC 500 m; CD 600 m. Construct a plan of this field with the aid of the scale in Fig. 192 and a protractor. Find from the scale the length of the sides DE and EA.

105. Indirect Measurement of Lengths. When we measure the length of a line on paper with a graduated ruler, or a distance on the ground with a chain, we are said to use *direct* measurement. But it often happens that direct measurement is inconvenient or even impossible, as in the case of the distance from the earth to the moon.

In these cases methods of *indirect* measurement are used. We measure directly certain lines and angles which are so related to the line to be measured that its length can be computed from our measurements.

Some problems on indirect measurement will now be considered.

The instruments in common use for measuring lengths are the **engineer's chain,** 100 ft. long, divided into 100 links; the **tape measure,** usually 50 ft. long, divided into feet and inches; the **Gunter's chain,** 66 ft. long, divided into 100 links.

For measuring angles, surveyors and engineers employ costly instruments called **transits** and **theodolites.** But in school work, a rough degree of accuracy will answer the purpose, and an angle on the ground may be immediately constructed without the aid of a protractor by the **pin-ruler** method (Fig. 193).

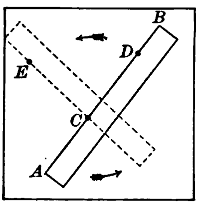

FIG. 193.

Suppose we wish to transfer to paper the angle formed at a certain spot by lines drawn, one to a house, the other to a tree. Fasten a sheet of paper to a small drawing board, and mount the board in a horizontal position directly over the given spot, using for this purpose a tripod and a spirit level. Stick a pin into the paper at C and hold the ruler against it. Let one person sight along the edge AB of the ruler, while another person holds a second pin against the edge and turns the ruler about C till the line of sight strikes the house.

Prick a hole D in the paper with the second pin. The line CD on the paper is one side of the required angle. Turn the ruler about C, still keeping the second pin touching the edge AB, till the line of sight

passes through the tree. Prick a hole E with the second pin. The angle DCE on the paper is the angle formed by the house and the tree.

106. Problem I. *To measure a line the ends of which only are accessible.*

Let AB (Fig. 194) be the line to be measured. Choose a point C from which A and B are visible. Measure AC and BC. Construct on paper by the pin-ruler method an angle $acb = ACB$ (Fig. 195). Lay off to a suitable scale of reduction the reduced lengths ac and bc of the lines AC and BC.

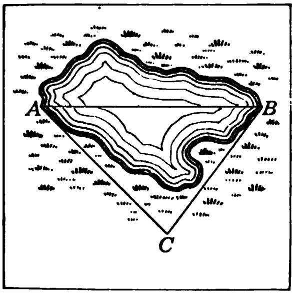

FIG. 194.

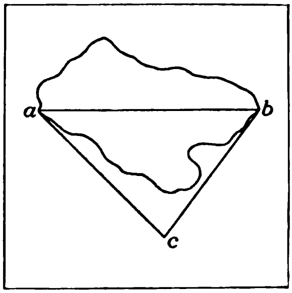

FIG. 195.

Draw ab and measure this line carefully. The length of AB corresponding to ab is the length required.

If the angle ACB is measured with a transit, an equal angle acb must be constructed on paper with a protractor.

NOTE. The use of a figure on paper may be avoided by constructing a triangle CDE on the ground similar to CAB. Extend AC to D and BC to E, each by the *same fraction* of its length.

EXERCISE

1. If AC (Fig. 194) = 1000 ft., $BC = 800$ ft., $ab = 6\frac{7}{16}$ in., and the scale is 1 in. to 200 ft., find AB.

107. Problem 2. *To measure a line of which one end only is accessible.*

Measure a straight line AC (Fig. 196) to a point C from which B is visible. Choose a scale of reduction, find the reduced length ac of AC, and draw ac on paper. Construct by the pin-ruler method the angles $bac = BAC$, and $bca = BCA$. Complete the triangle abc on the paper. Measure with care ab, and then find the length AB corresponding to ab.

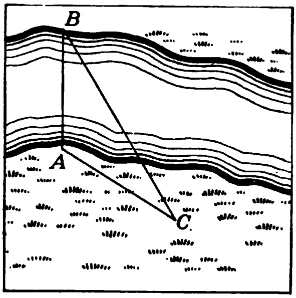

FIG. 196.

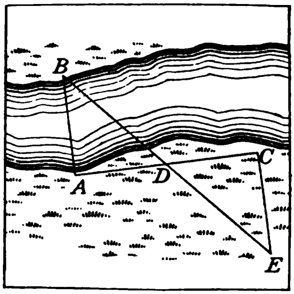

FIG. 197.

Another Method. The measurement of any other angles than right angles may be avoided by proceeding as follows: Starting at A (Fig. 197) lay off a straight line AC perpendicular to AB. Draw at C a perpendicular CE to AC. Bisect AC at D. Find on the line CE the point E, which is in the straight line BD extended. Measure CE. The length of CE is equal to the distance from A to B. For the right triangles ABD and CED are equal (p. 63).

EXERCISES

1. Find AB (Fig. 196) if $AC = 1200$ ft., $BAC = 100°$, $ACB = 50°$.

2. Explain why, if we lay off AD (Fig. 197) perpendicular to AB and of such length that the angle $ADB = 45°$, we know that $AB = AD$.

108. Problem 3. *To measure a line wholly inaccessible.*

Let it be required to find the length of the island AB (Fig. 198) by measurements made on the mainland. Choose a level place, lay off a straight line CD and measure its length. Lay off on paper the reduced length cd. Construct by the pin-ruler method the angles $acb = ACB$, $bcd = BCD$, $adb = ADB$, $adc = ADC$. Draw ab. Measure this line. The distance AB is the length corresponding to ab. The number of angles to be constructed in applying this method makes it difficult to secure a good result.

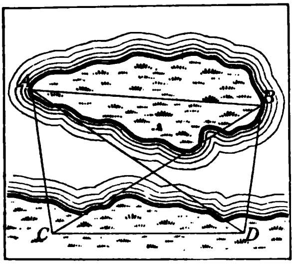

FIG. 198.

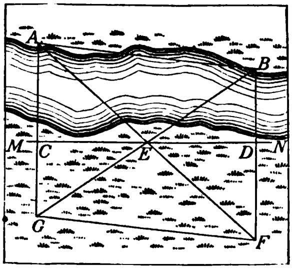

FIG. 199.

Another Method. Let it be required to measure the distance from A to B (Fig. 199) without crossing the river.

Lay off a straight line MN. Find on this line the points C and D through which perpendiculars from A and B, respectively, will pass. Bisect CD at E. Extend AE to meet the perpendicular BD at F, and extend BE to meet the perpendicular AC at G. Then $AC = DF$, $BD = CG$, and $AB = GF$.

Nothing remains, therefore, to be done but to measure GF.

In this method right angles only are used, and the laws of equal triangles are applied. What pairs of triangles in Fig. 199 are equal? Why are they equal (see p. 63)? What kind of a figure is $ABFG$?

109. Problem 4. *To measure the height of an object standing on a horizontal plane.*

Let AB (Fig. 200) be the object. From the foot A of the object measure in any direction a straight line AC. Place the instrument for measuring angles so that its centre shall be at D, directly above C, and its line of sight shall be horizontal, meeting AB at E. Measure DC and the angle EDB, which is called the *angle of elevation* of the point B. Draw to scale a right triangle edb similar to EDB. Measure eb and find EB, the length corresponding to the reduced length eb. Then $EB + DC = AB$.

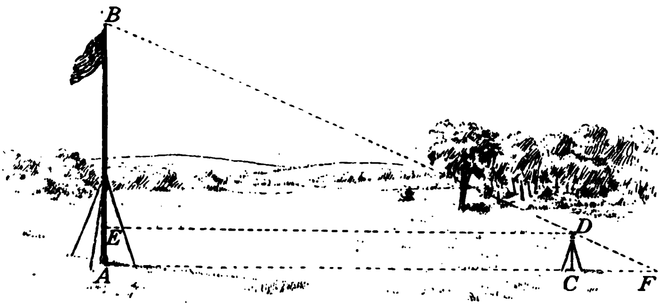

FIG. 200.

Another Method. Find the point F where BD extended meets AC extended. Measure AF, CF, and CD. Then, since the triangles AFB and CFD are similar, $CF : CD = AF : AB$.

Whence, $$AB = \frac{CD \times AF}{CF}.$$

EXERCISES

1. If $EDB = 30°$, $AC = 240$ ft., $DC = 4$ ft., find AB.
2. If $AC = 300$ ft., $CF = 10$ ft., $DC = 4$ ft., find AB.
3. How can this problem be solved by using the angle 45°?

110. Problem 5. *To find the height of an object by means of its shadow.*

When a vertical object casts a shadow on a horizontal plane, the length of the object, the length of the shadow, and the ray of light from the top of the object to the end of the shadow form a right triangle (Fig. 201). Moreover, since the sun is so far away that his rays are sensibly parallel lines, the right triangles formed at any instant by the sun shining on two vertical objects are similar. These facts are applied to find the height of an object.

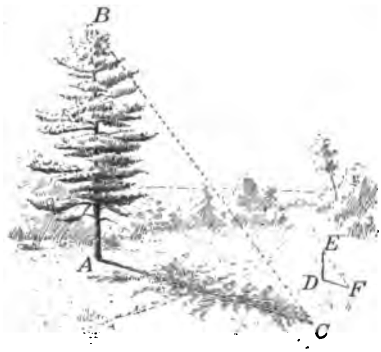

FIG. 201.

Suppose that a tree AB (Fig. 201) casts the shadow AC, and that at the same time a vertical pole DE casts the shadow DF. Measure AC, DE, DF. Then, since the triangles ABC and DEF are similar, $AB : DE = AC : DF$.

Whence, $$AB = \frac{AC \times DE}{DF}.$$

EXERCISES

1. If (Fig. 201) $AC = 60$ ft., $DE = 10$ ft., $DF = 8$ ft., find AB.

2. If $AC = 70$ ft., and the angle $DFE = 45°$, find AB.

3. The shadow of a church spire is 120 ft. long, and at the same time the shadow of a man 6 ft. tall is $3\frac{1}{2}$ ft. long. Find the height of the spire.

4. How high is a monument the shadow of which is 24 m long, when at the same time a meter rod casts a shadow 60 cm long?

5. If the shadow of a vertical rod is half as long as the rod, what is the height of a tree the shadow of which is 80 ft. long?

111. Problem 6. *To find the height of an inaccessible object.* Let it be required to find the height AB (Fig. 202) of a tree on the farther bank of a river without crossing the river. Find the point C directly opposite the tree where the angle of elevation ACB of the tree is 45°. Lay off from C a straight line CD perpendicular to AC and of such length that the angle $ADC = 45°$. Then we know that $AB = AC = CD$. All that remains to be done is to measure the length of CD.

FIG. 202.

Those who are familiar with the pin-ruler method of constructing angles (p. 126) will easily perceive how the angles ACB and ADC may be made equal to 45° by means of a square drawing board with its diagonals drawn.

EXERCISE

1. Wishing to find the height AB of a tree, I observe the angle of elevation ACB of the top B as seen from a point C, and find it to be 30°. At a point D on the line AC and 120 ft. nearer the tree I find the angle of elevation ADB to be 45°. Find the height AB of the tree.

112. Surveying. To **survey** a piece of land is to make such measurements as will determine its size and shape, to compute its area, and to draw to scale a plan of the land.

The only instruments required for making the measurements on the ground are a Gunter's chain for measuring lengths, and a square drawing board with its diagonals drawn for constructing right angles and running perpendiculars.

A Gunter's chain is 4 rods or 66 ft. long and contains 100 links. It has two advantages: first, the number of links can be written decimally as hundredths of a chain; secondly, square chains can be reduced to acres by dividing by 10.

1 acre $= 160$ sq. rods $= 10 \times 16$ sq. rods $= 10$ sq. chains.

Problem 1. *To survey the open four-sided field ABCD.*

Begin by marking out and measuring the diagonal BD (Fig. 203).

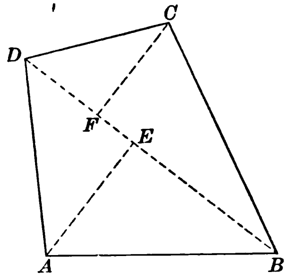

FIG. 203.

Then determine the points E, F on BD, where perpendiculars from the points A and C, respectively, meet BD. Then measure the lengths of AE and CF. Then find the areas of the triangles ABD and CBD. Their sum will be the area of the field. Lastly, draw to scale a plan of the field, beginning with the diagonal BD.

EXERCISE

1. Find the area and construct to a suitable scale a plan of the field $ABCD$ from the following measurements:

$BD = 14.36$ chains, $AE = 8.17$ chains, $CF = 5.74$ chains.

Problem 2. *To survey a field ABCD when its diagonals cannot conveniently be measured.*

Through one corner A of the field run a straight line $EAGF$. Find the points E, G, F where perpendiculars from the corners B, C, and D, respectively, meet the line. Measure AE, BE, AF, CF, AG, and DG. Find the areas of the trapezoids $EBCF$ and $GDCF$; and of the triangles AEB and AGD. Find the area of the field, and draw to scale a plan of the field, erasing all the auxiliary lines.

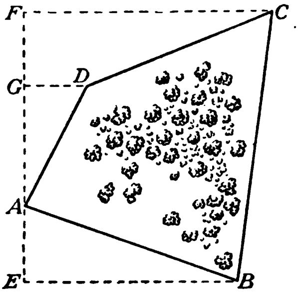

FIG. 204.

EXERCISES

1. Find the area of the field in Fig. 205, having given

$AF = 12$ chains, $BF = 7$ chains, $AG = 8$ chains, $CG = 12$ chains,
$AK = 3$ chains, $DK = 12$ chains, $AL = 8$ chains, $EL = 5$ chains.

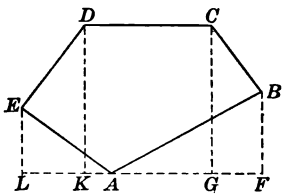

FIG. 205.

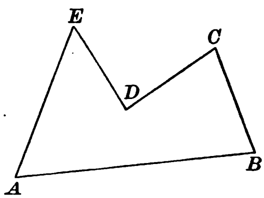

FIG. 206.

2. Describe how you would proceed in order to find the area of a field having the shape $ABCDE$ (Fig. 206). Begin by drawing a figure similar to $ABCDE$. Then put in the lines which you would measure in order to obtain the area. Then describe clearly how you would compute the area.

Problem 3. *To survey an inaccessible field ABCD by means of an auxiliary figure.*

Sometimes the direct measurement of straight lines in a field is difficult or impossible by reason of obstacles such as woods, swampy land, etc. One way of meeting this difficulty is illustrated in Fig. 207.

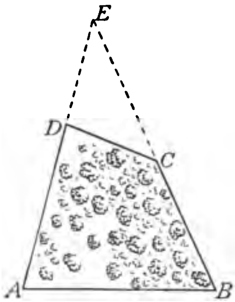

FIG. 207.

First, measure the lengths of the sides AB, BC, CD, and DA. Then extend the sides AD and BC till they meet at E. Then draw to scale a plan dce of the triangle DCE. Extend on the paper ed to a point a such that the length ad on the paper shall correctly represent, to the scale used, the length AD on the ground. Extend in the same way ec to a point b so that bc on the paper shall represent BC on the ground. Draw ab. The figure $abcd$ on the paper is a plan of the field $ABCD$.

To find the area of the field, first find the area of the figure $abcd$, then multiply this area by the **square** of the number of chains represented by 1 unit of length on the plan; the product will be the area of the field in square chains.

For example, suppose that the scale of reduction is 1 cm to 3 chains. Then 1 qcm of area on the plan represents an actual area on the field of 9 sq. chains. If the area of $abcd$ is 12.27 qcm, then the area of the field will be $12.27 \times 9 = 110.43$ sq. chains $= 11.043$ acres.

EXERCISE

1. If (Fig. 207) $AB = 24$ chains, $BC = 12$ chains, $DC = 10.58$ chains, $AD = 16$ chains, $DE = 8$ chains, $CE = 12$ chains, construct a plan of the field and find its area.

Another Method. Lay out a rectangle which completely encloses the field, and draw perpendiculars from all the corners of the field to the sides of the rectangle. Subtract the sum of the areas of the right triangles and trapezoids thus formed from the area of the rectangle, the remainder will be equal to the area of the field.

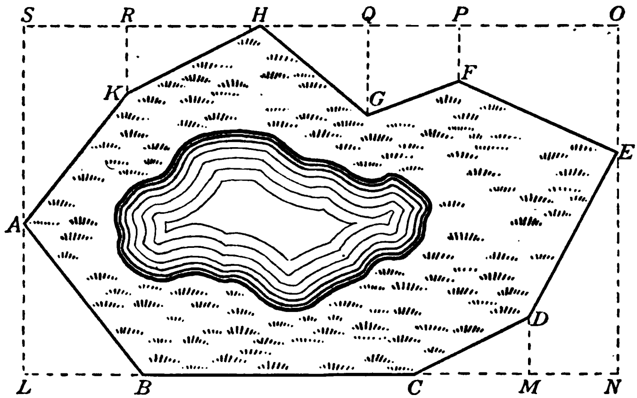

FIG. 208.

EXERCISE

Find the area of the polygon $ABCDEFGHK$, having given

$SA = 30$ chains	$MD = 10$ chains	$QH = 20$ chains
$AL = 25$ "	$NE = 35$ "	$HR = 22$ "
$LB = 20$ "	$EO = 20$ "	$SR = 18$ "
$BC = 45$ "	$OP = 25$ "	$PF = 10$ "
$CM = 20$ "	$PQ = 15$ "	$QG = 15$ "
$MN = 15$ "		$RK = 12$ "

Arrange the results of computing the areas in neat form as follows:

Rectangle $SLNO$ $= 55 \times 100 = 5500$ sq. chains $= 550$ acres.
Right triangle $ALB = 25 \times 10 = 250$ " " $= 25$ "

CHAPTER VII

THE COMMON GEOMETRIC SOLIDS

113. The Right Prism. A solid bounded by two equal polygons in parallel planes and by three or more rectangles is called a **right prism** (Fig. 209).

The two polygons are called the **bases**, the distance between the bases is called the **height** or **altitude**, the rectangles are called the **lateral faces**, the intersections of the lateral faces are called the **lateral edges**.

A right prism is said to be **triangular**, **quadrangular**, etc., according as its bases are triangles, quadrilaterals, etc.

FIG. 209.

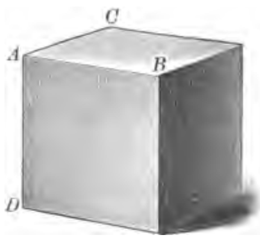

FIG. 210.

A right prism, the bases of which are regular polygons, is called a **regular prism**.

A right prism, the bases of which are rectangles (Fig. 209), is called a **rectangular parallelopiped**.

A rectangular parallelopiped may also be defined as a solid bounded by six rectangles.

A **cube** is a solid bounded by six equal squares (Fig. 210).

114. Parallel Perspective. To make a drawing of a cube like that in Fig. 211, begin by drawing the square $ABCD$.

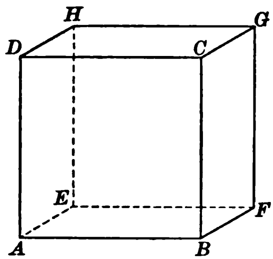

FIG. 211.

Then draw AE making the angle BAE equal to $30°$ and AE equal to $\frac{1}{3}AB$. Then draw BF, CG, and DH parallel and equal to AE.

Draw EF, HG, EH, and FG.

The edges of the cube are now all represented on the paper, and the diagram is complete. The edges shown with dashes, AE, EF, EH, are the edges which would be invisible if the cube were made of an opaque material such as wood.

In this diagram the face $ABCD$ is to be regarded as lying in the plane of the paper, the face $EFGH$ as parallel, and the other faces as perpendicular to the plane of the paper.

The eight edges which lie either in the plane of the paper or parallel to it are drawn in their true relative directions and equal in length. The remaining edges, AE, BF, CG, DH, must be so drawn that they shall appear to be perpendicular to the plane of the paper and equal to the other edges; the method of drawing them just described is one way of meeting these conditions.

The result is that only two faces of the cube are drawn as squares, the other faces being parallelograms.

A diagram of a cube made as here described is called a **parallel projection** or **parallel perspective** of a cube.

It is not necessary that the angle BAE in Fig. 211 should be made equal to exactly $30°$, or that AE should be made equal to exactly $\frac{1}{3}AB$. Other values for the angle of reduction and the reducing ratio, within certain limits, are allowable. But $30°$ and $\frac{1}{3}$ are good values for the cube.

Parallel perspective is one of the ways of representing on paper objects that have three dimensions in space. It does not reproduce the exact appearance of an object as seen from any point of view. It serves, however, to suggest to the mind the true relations of the parts of the common geometric solids better than any other method. All lines in space, for example, that are parallel and equal are represented in the diagram by lines that are parallel and equal.

EXERCISES

1. Make a drawing in parallel perspective of a right prism with a square base and lateral edges twice as long as one side of the base.

This differs from the cube only in the length of the lateral edges.

2. Put in parallel perspective a right parallelopiped, taking as the dimensions: length 12 cm, width 6 cm, height 9 cm. In Fig. 212 the front edge of the base represents the length of the parallelopiped, and the width runs from the front backward. In drawing the edges that are perpendicular to the plane of the paper use the angle of reduction 30°, and the reducing ratio $\frac{1}{3}$, as in the case of the cube.

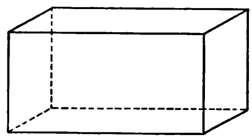

FIG. 212.

3. Put in parallel perspective the rectangular parallelopiped of Ex. 2 so that the length shall extend from the front backwards.

4. Put in parallel perspective the rectangular parallelopiped of Ex. 2 with the height and width interchanged.

5. Draw a rectangular parallelopiped, taking as the dimensions: length 5 in., width 3 in., height 4 in.

6. Draw a rectangular parallelopiped, taking as the dimensions: length 8 cm, width 4 cm, height 10 cm.

7. The dimensions of a right prism are: length 8 cm, width 6 cm, height 10 cm. Put the prism in parallel perspective,

(1) using the angle of reduction 30° and the reducing ratio $\frac{1}{3}$;
(2) using the angle of reduction 45° and the reducing ratio $\frac{1}{2}$;
(3) using the angle of reduction 60° and the reducing ratio $\frac{1}{3}$.

8. Put in parallel perspective a regular triangular prism, making its height one and a half times the length of one side of its base.

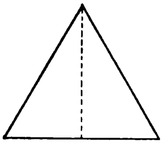

FIG. 213.

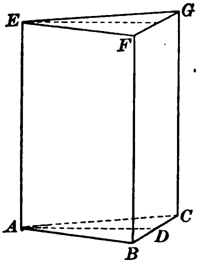

FIG. 214.

Begin by drawing an equilateral triangle and its altitude (Fig. 213). Then draw a horizontal line AD equal to the altitude of the triangle. Through D draw BC so that $ADB = 30°$, $BC = \frac{1}{2} AD$, and D is the middle point of BC. Draw AB and AC. ABC is the base of the prism. Then draw AE perpendicular to AD, and equal to $\frac{3}{2}$ of one side of the equilateral triangle. Draw BF and CG equal and parallel to AE. Draw EF, FG, GE.

9. Draw a regular hexagonal prism, making its height three times as long as one side of the base.

Draw a regular hexagon $ABCDEF$ and the diagonals FC, AE, BD. Then draw a horizontal line MN equal to FC. On this line lay off MO equal to FG, and NP equal to CH. Through O draw QR equal to $\frac{1}{3}$ AE, making the angle 30° with MN, and bisected at the point O. Through P draw ST parallel and equal to QR, and bisected at the point P. The figure $MQSNTR$ is the base of the prism. Then draw all the lateral edges of the hexagon perpendicular to MN, and make each one three times as long as QS. Connect by straight lines the upper ends of the lateral edges, and the figure is complete.

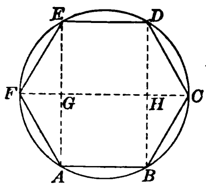

FIG. 215.

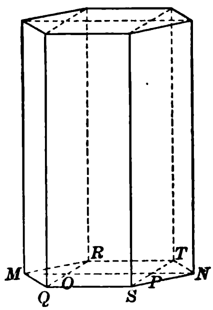

FIG. 216.

10. Put in parallel perspective the regular hexagonal prism of Ex. 9, using the angle of reduction 45° and the reducing ratio $\frac{1}{2}$.

11. Put in parallel perspective a right prism having for base an isosceles triangle with sides 4 cm, 8 cm, 8 cm, and having an altitude of 12 cm.

115. The Regular Pyramid. A solid bounded by a regular polygon and three or more isosceles triangles having a common point for vertex is called a **regular pyramid** (Fig. 217).

The regular polygon is called the **base** of the pyramid.

The triangles are called the **lateral faces** of the pyramid, and their intersections are called the **lateral edges.**

The common vertex of the triangles is called the **vertex** of the pyramid.

The lateral edges are equal lines, and the lateral faces are equal triangles.

A pyramid is called **triangular, quadrangular,** etc., according as its base is a triangle, a quadrilateral, etc.

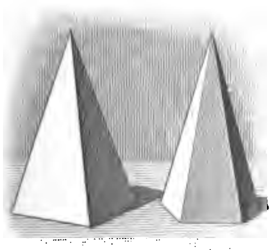

FIG. 217.

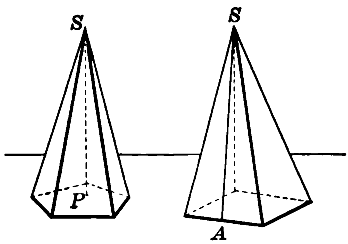

FIG. 218.

The **height** or **altitude** of a pyramid is the length of the perpendicular drawn from the vertex to the base.

In a regular pyramid this perpendicular passes through the centre of the base, and is also called the **axis** of the pyramid (*SP*, Fig. 218).

The **slant height** of a regular pyramid is the altitude of any one of the lateral faces. The slant height (*SA*, Fig. 218) bisects the side of the base to which it is drawn. The slant height is always greater than the height of the pyramid.

EXERCISES

1. Put in parallel perspective a regular pyramid with a square base.

The square base $ABCD$ (Fig. 219) is put in parallel perspective by proceeding as already explained in the case of the cube, using the reducing ratio $\frac{1}{2}$ instead of $\frac{1}{3}$.

The axis of the pyramid is then drawn perpendicular to AB from the point O, the intersection of the diagonals AC and BD. From V, any point in the axis, as the vertex of the pyramid, draw straight lines to the points A, B, C, and D.

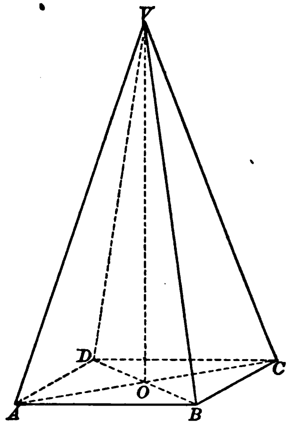

FIG. 219.

2. Put in parallel perspective a regular pyramid with a square base, having dimensions as follows: side of base 6 cm, height of pyramid 10 cm.

3. Put in parallel perspective a regular triangular pyramid, the height of which is twice one side of the base.

Construct an equilateral triangle (Fig. 220). Draw two of its altitudes. The intersection of these altitudes is the centre of the triangle, and divides each altitude into two parts such that the smaller part is $\frac{1}{3}$ of the whole altitude.

Draw the equilateral triangle in parallel perspective as explained in Ex. 8, p. 140. The resulting figure ABC is the base of the pyramid. The foot O of the axis of the pyramid is found by laying off from D, on the altitude, $DO = \frac{1}{3}AD$.

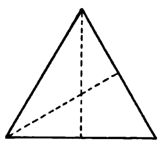

FIG. 220.

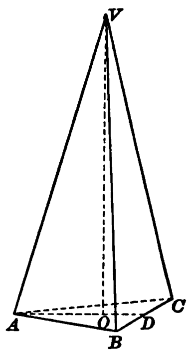

FIG. 221.

4. Put in parallel perspective a regular hexagonal pyramid.

The hexagonal base of the pyramid is drawn exactly as explained in the case of an hexagonal prism (Ex. 9, p. 140).

116. Cylinder of Revolution. The solid generated by a rectangle revolving about one of its sides as an axis is called **a cylinder of revolution** (Fig. 222).

The side AB about which the rectangle revolves is the **axis** of the cylinder; its length is the **height** of the cylinder.

The opposite side CD generates a curved surface called the **lateral surface** of the cylinder.

The sides AD and BC generate circles called the **bases** of the cylinder.

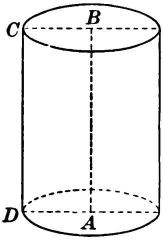

FIG. 222.

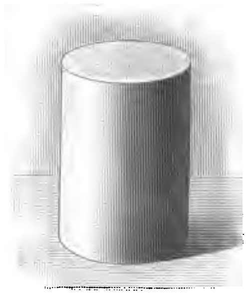

FIG. 223.

A cylinder of revolution is also called a **right circular cylinder.**

EXERCISE

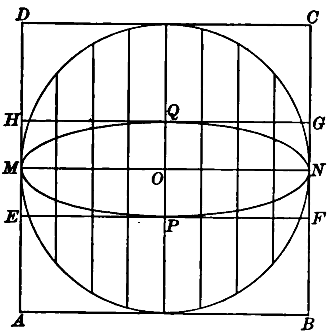

FIG. 224.

1. Draw a cylinder of revolution. If the axis of the cylinder is vertical, the bases will appear as ellipses.

An ellipse is drawn with the reducing ratio $\frac{1}{3}$ and without change of angles, as shown in Fig. 224. The square $ABCD$ becomes the rectangle $EFGH$, and the inscribed circle becomes an ellipse inscribed in the rectangle and having for diameters MN and PQ. Points on the ellipse are found by drawing chords perpendicular to MN and reducing each chord.

In drawing the base of a cylinder the square and rectangle may be omitted. After the points on the ellipse have been found, the ellipse is drawn free-hand through these points.

117. Cone of Revolution. The solid generated by a right triangle revolving about one of its legs as an axis is called **a cone of revolution** (Fig. 225).

The leg AB about which the triangle revolves is the **axis** of the cone, its length the **height** or **altitude** of the cone.

The hypotenuse BC generates a curved surface called the **lateral surface** of the cone.

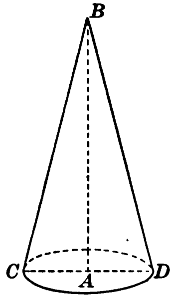

Fig. 225.

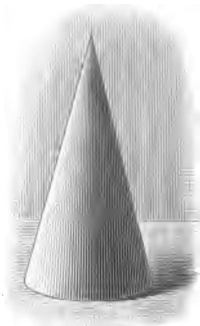

Fig. 226.

The leg AC generates a circle called the **base** of the cone.

The length of the hypotenuse BC is called the **slant height** of the cone.

A cone of revolution is also called **a right circular cone.**

EXERCISES

1. Draw a cone of revolution.

Begin by putting a circle in parallel perspective, and the resulting ellipse is the base of the cone. Erect at the centre of the ellipse a line, for the axis of the cone.

2. The height of a cone of revolution is 12 cm, the radius of its base is 5 cm. Find its slant height (see p. 104).

3. The slant height of a cone of revolution is 20 ft., the radius of its base is 12 ft. Find the height of the cone.

118. Frustums. The **frustum** of a pyramid or of a cone is the portion of the pyramid or the cone included between the base and a plane parallel to the base (Figs. 227 and 228).

The base and the section parallel to the base are called the **bases** of the frustum. The **height** or **altitude** of the frustum is the distance between its bases.

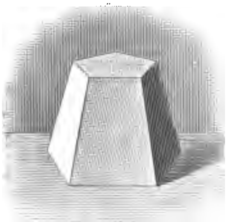

FIG. 227.

FIG. 228.

In the frustum of a regular pyramid the two sides of the bases contained between the same two parallel edges are parallel lines; the lateral faces are equal isosceles trapezoids; the **slant height** is the altitude of one of these trapezoids.

In the frustum of a cone of revolution that portion of the slant height of the cone included between the bases of the frustum is called the **slant height** of the frustum.

EXERCISES

1. Draw the frustum of a regular pyramid with a square base.

Draw the entire pyramid. Put in the upper base of the frustum by drawing each one of its sides parallel to the corresponding side of the lower base. Erase the part above the upper base.

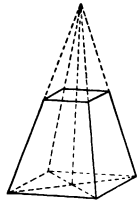

FIG. 229.

2. Draw the frustum of a regular hexagonal pyramid.

Hint. Begin by drawing the complete pyramid.

3. Draw the frustum of a cone of revolution.

Hint. Begin by drawing the complete cone.

119. The Sphere. A semicircle revolving about its diameter generates a solid called a **sphere** (Fig. 231).

The meaning of the terms **centre, radius, diameter,** as applied to a sphere, will be plain from their definitions as applied to a circle.

Every section of a sphere made by a plane is a circle.

The circle is called a **great circle** if the plane passes through the centre, a **small circle** in all other cases (Fig. 232).

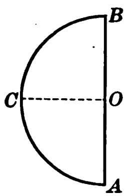

FIG. 230.

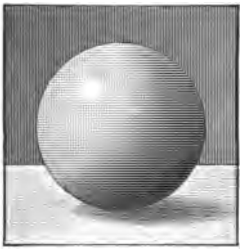

FIG. 231.

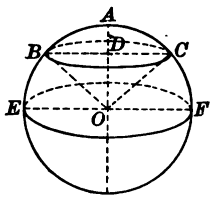

FIG. 232.

The part of a sphere contained between two parallel planes is called a **spherical segment** (Fig. 232), and the part of the surface of the sphere contained between the planes is called a **zone.** The circular sections made by the planes are called the **bases** of the segment, and the distance between them is called the **height** of the segment or zone.

A single plane divides a sphere into two parts called **segments of one base,** and the surface into two parts called **zones of one base.** A great circle divides a sphere into two equal segments called **hemispheres.**

The portion of a sphere generated by the revolution of a circular sector *AOB* (Fig. 232) about one of its radii is called a **spherical sector.**

120. Surfaces of the Common Solids. The most important cases are covered by the following formulas, in which S denotes lateral area or the area of curved surface:

Right Prism.

$$S = \textit{perimeter of base} \times \textit{altitude}.$$

Cylinder of Revolution.

$$S = \textit{circumference of base} \times \textit{altitude}.$$

Regular Pyramid.

$$S = \textit{perimeter of base} \times \tfrac{1}{2}\ \textit{slant height}.$$

Cone of Revolution.

$$S = \textit{circumference of base} \times \tfrac{1}{2}\ \textit{slant height}.$$

Frustum of Regular Pyramid.

$$S = \tfrac{1}{2}\ \textit{sum of perimeters of bases} \times \textit{slant height}.$$

Frustum of Cone of Revolution.

$$S = \tfrac{1}{2}\ \textit{sum of circumferences of bases} \times \textit{slant height}.$$

Sphere.

Let r = radius of sphere, h = altitude of zone, and π denote the number $\frac{22}{7}$, or 3.1416.

$$\textit{Area of surface of sphere} = 4\,\pi r^2.$$

$$\textit{Area of zone} = 2\,\pi rh.$$

EXERCISES

1. What is the entire surface of a cube whose edge is 6 cm long?

2. How many square feet of lead are required to line the bottom and sides of a cubical tank $4\frac{1}{2}$ ft. deep?

3. Find the entire surface of a rectangular parallelopiped if the length $= 15$ cm, the breadth $= 9$ cm, and the depth $= 6$ cm.

4. A marble pillar 12 ft. high has a square base one edge of which is 2 ft. 4 in. long. What will it cost to polish the lateral surface of the pillar at 50 cents per square foot?

5. How many bricks 8 in. by $3\frac{1}{2}$ in. are required to line the bottom and sides of a reservoir 60 ft. long, 24 ft. wide, and 12 ft. deep?

6. The height of a regular hexagonal prism is 32 ft. One side of its base is 3 ft. Find its entire surface (see p. 102).

7. Find the entire surface of a right circular cylinder 10 in. high, the radius of the base being 5 in.

8. How many square feet of tin are required to cover the lateral surface of a circular tower 40 ft. high, the diameter of the base being 8 ft.?

9. Find the cost of cementing the bottom and side of a cylindrical tank 20 ft. deep and 18 ft. in diameter at 30 cents per square foot.

Find the lateral surface of a regular hexagonal pyramid, given:

10. Edge of base 20 cm, slant height 30 cm.

11. Edge of base 10 cm, lateral edge 13 cm (see p. 104).

12. Edge of base 8 cm, height of pyramid 25 cm.

13. How many square feet of tin are required to cover the lateral surface of a square pyramid 16 ft. high and side of base 8 ft.?

14. The height of a right cone is 16 cm and the radius of its base is 6 cm. Find its slant height and lateral surface.

15. A right triangle whose legs are 3 in. and 4 in. generates a cone by revolving about its shorter leg. Find the entire surface of this cone.

16. How much canvas is required to make a conical tent 80 ft. high and 70 ft. in diameter at the base?

17. Find the total surface of a frustum of a square pyramid if the sides of its bases are 12 in. and 4 in., and the slant height is 5 in.

18. Find the height of the frustum in Ex. 17.

19. A frustum of a regular triangular pyramid is 10 ft. high, and the sides of its bases are 40 ft. and 20 ft. Find its lateral surface.

20. A church spire has the shape of a frustum of a regular hexagonal pyramid. Each side of the lower base is 10 ft. long, and each side of the upper base is 4 ft. long. The slant height of the frustum is 80 ft. How many square feet of tin are required to cover the lateral faces and the top?

21. Find the entire surface of the frustum of a right cone, given the radii of its bases 14 cm and 9 cm, and its height 12 cm.

22. Find the surface of a sphere whose diameter is 7 in.

23. Find the surface of a sphere whose circumference is 22 ft.

24. The radius of a sphere is 35 ft., and the altitude of a zone is 15 ft. Find the area of the zone.

25. A baseball is $9\frac{1}{4}$ in. in circumference. How much leather is required to cover 1000 baseballs?

26. The circumference of a dome in the shape of a hemisphere is 66 ft. How many square feet of tin roofing are required to cover it?

27. If the ball on the top of St. Paul's Cathedral in London is 6 ft. in diameter, what will it cost to gild it at 7 cents per square inch?

28. The altitude of the torrid zone on the earth is about 3200 miles. What is its area in square miles, assuming the radius of the earth to be 4000 miles?

29. The edges of two cubes are to each other as 2 : 1. Compare their surfaces.

30. If each dimension of a rectangular parallelopiped is multiplied by 3, what effect is produced on the surface?

31. The heights of two circular cylinders having equal bases are as 2 : 1. Compare their lateral surfaces.

32. The radii of the bases of two cylinders having equal heights are as 2 : 1. Compare their lateral surfaces.

33. If the height and the radius of the base of a cylinder are both doubled, what is the effect on the lateral surface?

34. Compare the lateral surfaces of a cylinder and a cone having equal bases and equal altitudes.

35. The lateral surfaces of a cylinder and a cone having equal bases are equal. Compare their altitudes.

36. The radii of two spheres are as 4 : 1. Compare the surfaces.

37. The radius of a sphere is 7 in. The radius of the base of a cone is also 7 in. What must be the height of the cone in order that its lateral surface may be equal to the surface of the sphere?

121. Volumes of the Common Solids. A **unit of volume** is a cube whose edge is equal to a unit of length.

The **volume** of a solid is the number of units of volume it contains.

The most important units of volume are :

The cubic inch (cu. in.).
The cubic foot (cu. ft.) $=1728$ cu. in.
The cubic yard (cu. yd.) $=27$ cu. ft.
The cubic centimeter (ccm).
The cubic decimeter (cdm), or liter (l) $=1000$ ccm.
The cubic meter (cbm) $=1000$ cdm.

The volume V of a solid can be found in the most important cases by applying the following formulas :

Rectangular Parallelopiped.

$$V = \textit{length} \times \textit{breadth} \times \textit{height}.$$

Right Prism or Cylinder of Revolution.

$$V = \textit{base} \times \textit{altitude}.$$

Regular Pyramid or Cone of Revolution.

$$V = \tfrac{1}{3} \times \textit{base} \times \textit{altitude}.$$

Any Frustum (*B* and *b* its bases).

$$V = \tfrac{1}{3} \times \textit{altitude} \times (B + b + \sqrt{B \times b}).$$

Sphere.

$$V = \tfrac{4}{3}\pi \times \textit{cube of radius}.$$

EXERCISES

A cubic foot of water weighs 62.4 lb.

1. What is the volume of a cube if each edge is 8 in. long?

2. Find one edge of a cube if the volume is 64 cu. ft.

3. Each edge of a cubical tank measures 5 ft. 6 in. How many cubic feet of water will it hold? What is the weight of the water?

4. How many tons of coal will a bin 20 ft. by 16 ft. by 8 ft. hold, allowing 40 cu. ft. to a ton?

5. A cellar, the floor of which measures 12 ft. by 6 ft., is flooded to the depth of 8 in. Find the weight of the water.

6. How many bricks 9 in. by $4\frac{1}{2}$ in. by 3 in. are required to build a wall 90 ft. long, 8 ft. high, and 18 in. thick?

7. A book is 8 in. long, 6 in. wide, and $1\frac{1}{4}$ in. thick. A box is 3 ft. 4 in. long and 2 ft. 6 in. wide. How deep must the box be to hold 400 books?

8. Find the weight of a block of marble 9 ft. 6 in. long, 2 ft. 3 in. wide, and 2 ft. thick, if marble is 2.7 times as heavy as water.

9. Find the volume of a regular triangular prism if the height of the prism is 7 ft. and one side of the base is 2 ft.

10. Find the volume of a prism 10 ft. high the bases of which are regular hexagons each having a side 10 in. long.

11. How many cubic yards of earth must be excavated in making a cylindrical well 3 ft. in diameter and 20 ft. deep? What weight of water will the well contain when full?

12. What is the cost of making a tunnel 100 yd. long whose section is a semicircle with a radius of 10 ft. at $6 per cubic yard of material removed?

13. What change in the volume of a cylinder is produced by doubling its height? by doubling the diameter of its base? by doubling both?

14. A rectangular sheet of tin is 88 in. long and 66 in. wide. Find the volume of the cylinder made by rolling up the sheet:

    (1) so that the height of the cylinder is 88 in.;

    (2) so that the height of the cylinder is 66 in.

15. Find the volume of a square pyramid if a side of the base is 40 ft. and the height is 48 ft.

16. Find the volume of a pyramid if the height is 30 ft. and the base is a regular hexagon whose side is 6 ft.

17. Find the volume of a cone if the height is 28 ft. and the radius of the base is 14 ft.

18. A right triangle whose legs are 6 in. and 8 in. long is revolved about the shorter leg. Find the volume of the cone thus generated.

19. Find the volume of the frustum of a square pyramid if the sides of the bases are 12 in. and 4 in. and the height is 6 in.

20. The sides of the bases of the frustum of a square pyramid are 16 ft. and 32 ft. The *slant* height is 20 ft. Find the volume.

21. A round stick of timber is 20 ft. long, 3 ft. in diameter at one end and 2 ft. 6 in. at the other. How many cubic feet of wood are there in the stick?

22. A bucket is 16 in. deep, 18 in. in diameter at the top, and 12 in. in diameter at the bottom. How many gallons of water will it hold, reckoning $7\frac{1}{2}$ gal. to the cubic foot?

23. Find the volume of a sphere if the diameter is 28 ft.

24. Find the volume of a sphere if the radius is 3 ft. 6 in.

25. If one cubic inch of iron weighs a quarter of a pound, what will an iron ball 1 ft. in diameter weigh?

26. The piston of a pump is 14 in. in diameter and moves through a space of 3 ft. How many tons of water will be thrown out by 1000 strokes?

27. A body is placed under water in a right cylinder 60 in. in diameter, and the level of the water is observed to rise 30 in. Find the volume of the body.

28. Compare the volumes of a cylinder and a cone that have equal bases and equal heights.

29. The edge of a cube is 10 in. Find:

(1) the volume of the largest cylinder that can be made from it;
(2) the volume of the largest cone;
(3) the volume of the largest sphere.

30. The diameter of a wooden cylinder and its height are each 14 in. Find:

(1) the volume of the cylinder;
(2) the volume of the largest sphere that can be made from the cylinder.

ANSWERS

Page 43. **7.** 360. **8.** 5400. **9.** 36,380″; 5°. **10.** 150°; 135°; 120°; 100°; 60°; 30°. **11.** 80°; 70°; 60°; 50°; 30°; 15°. **12.** 90°. **13.** 45°. **14.** 60°.

Page 54. **3.** $c = e = g = 40°$; $b = d = f = h = 140°$. **4.** 90°. **5.** $c = e = g = 60°$; $b = d = f = h = 120°$.

Page 55. **5.** $4\frac{1}{4}$ mi. nearly. **6.** $6\frac{5}{8}$ mi. nearly. **7.** 144° 37′ 30″. **8.** 10° 38′. **9.** 93° 15′. **10.** 33° 54′. **11.** 51° 25′ 43″. **12.** 51° 25′ 43″.

Page 60. **2.** 150°. **3.** 80°. **4.** 50°.

Page 61. **6.** 45°. **8.** 100°. **9.** 79° 6′.

Page 62. **1.** 130°. **2.** 90°. **3.** 90°. **5.** 114°.

Page 63. **2.** 40°; 100°. **3.** 65°; 65°. **4.** 45°.

Page 64. **5.** 60°. **6.** 60°; 30°. **7.** 40°; 140°; 140°.

Page 67. **2.** $22\frac{1}{2}$. **1.** 30°; 60°.

Page 82. **1.** 720°. **2.** 1080°. **3.** 1440°. **4.** 1800°. **5.** 9.

Page 84. **2.** 120°; 90°; 72°; 60°; 45°; 36°; 3° 36′. **3.** 120°. **4.** 135°. **5.** 144°.

Page 92. **1.** 110 ft. **2.** 88 in. **3.** 50.2656 cm. **4.** 42 ft. **5.** 0.15915 m. **6.** 66,080. **7.** $668\frac{2}{11}$ ft. **8.** 7 ft. $7\frac{7}{11}$ in. **9.** $1536\frac{32}{63}$ ft. **10.** $2\frac{19}{22}$ in. **11.** 3 ft. 8 in. **12.** 11 in. **13.** 7 ft. $7\frac{2}{3}$ in. **15.** 45°. **16.** 1 mi.; 57° 16′ 22″. **17.** 57° 16′ 22″.

Page 95. **1.** 81 sq. in. **2.** 196 sq. ft. **3.** $4\frac{33}{64}$ sq. in. **4.** 225 qcm. **5.** 625 qm. **6.** 41.2164 qcm. **7.** $68.45. **8.** 285,156 sq. ft.; 6.546 A.

Page 96. **1.** 108 sq. in. **2.** 93 sq. in. **3.** $7\frac{7}{8}$ sq. in. **4.** 12,000 sq. ft. **5.** 6000 sq. ft. **6.** 192 qcm. **7.** 800 qcm. **8.** 5 in. **9.** 12 in. **10.** $2\frac{1}{2}$ in. **11.** 10 ft. **13.** $5866\frac{2}{3}$ sq. yd.

Page 97. **14.** 2312. **15.** 280 sq. ft.; 40; 40 yd. **16.** $4233.60. **17.** 2 ft. 8 in. **18.** 100 rd. **19.** Doubled; doubled; quadrupled. **20.** 112,320 sq. ft. **21.** Square by 25 qm. **22.** 6 in. **23.** 6.32 in. **24.** 7650 sq. ft.; 625 sq. ft. **25.** 400 sq. ft.

Page 98. **1.** 80 sq. in. **2.** 35 sq. in. **3.** 54.94 qcm.

Page 99. **1.** 55 sq. in. **2.** $1\frac{1}{4}$ sq. in. **3.** $105\frac{1}{12}$ sq. ft. **4.** 44.72 qcm.

Page 100. **1.** 24 sq. in. **2.** $29\frac{19}{64}$ sq. in. **3.** 21,780 sq. ft. **4.** 360 ft. **6.** 24 sq. in.

Page 101. **1.** 54 sq. in. **2.** 125 qcm. **3.** 589 sq. ft. **4.** $\frac{1}{8}$ sq. mi. **5.** 560 sq. ft. **6.** 138,564 sq. ft.

Page 102. **1.** 10.8 sq. in. **2.** 61.92 sq. in. **3.** 1086.3 sq. ft. **4.** 2777.895 sq. ft. **5.** $207,840. **6.** 110.848 sq. ft. **7.** 192 sq. ft. **8.** 1.61 ft.

Page 103. **1.** 8 A. 1280 sq. yd. **2.** 1 A. 3104 sq. yd. **3.** 20 A. 4750 sq. yd. **4.** 10 A. 391 sq. yd.

Page 105. **1.** 10 cm. **2.** 17 cm. **3.** 12 cm. **4.** 12 cm. **5.** 30 ft. **6.** 26 ft. **7.** 1200 ft. **8.** 60 mi. **9.** 150 sq. in. **10.** 28.28 cm. **11.** 14.14 cm. **12.** 48 sq. ft. **13.** 17.32 ft.; 173.2 sq. ft. **14.** 28.28 ft. **15.** 14.14 ft. **16.** 30 ft.

Page 106. **1.** 154 sq. in. **2.** 616 sq. ft. **3.** 1256.64 qcm. **4.** 3.1416 qm. **5.** 1886.5 sq. ft. **6.** 9.625 sq. ft. **7.** 201.0624 qcm. **8.** $7857\frac{1}{7}$ sq. in. **9.** 125.714 A. **10.** 5544 sq. ft. **11.** 770 sq. ft.

Page 107. **12.** 3080 sq. ft. **13.** $400. **14.** 16. **15.** 616 sq. in. **18.** 576.72 ft. **19.** 3.74 in. **20.** 6.2 ft. **21.** 5.66 in.; 8 in. **22.** 74.45 ft. **23.** 7.90 ft. **24.** 50.91 A. **25.** $207\frac{3}{7}$ yd.; $204\frac{2}{7}$ sq. yd.

Page 108. **26.** 77 sq. in. **27.** $12\frac{5}{6}$ sq. in. **28.** $1283\frac{1}{3}$ sq. in. **29.** $102\frac{2}{3}$ sq. ft. **30.** 462 sq. in. **31.** 77 sq. in.; 45°. **32.** $\frac{1}{6}$; 60°. **33.** 4.45 sq. in. **34.** 896 sq. ft. **35.** 4.5 sq. ft.

Page 110. **1.** 1 : 2 ; 2 : 1 ; 2 : 3 ; 1 : 3 ; 3 : 7 ; 7 : 5.

Page 111. **1.** 1 : 4 ; 8 : 1 ; 1 : 6. **3.** 40. **4.** 15. **5.** 4. **6.** 10. **7.** 1.5. **8.** 9.

Page 112. **2.** 320 sq. ft. **3.** 33 : 13 ; 33 : 20. **4.** 300 sq. ft. **8.** 756 sq. ft.; 24 ft.

Page 113. **11.** 10 : 1. **12.** 1 : 2. **13.** 2 : 1. **14.** 2 : 1. **19.** 1 : 3 ; 1 : 9. **20.** Doubled; quadrupled. **21.** 1 : 5 ; 1 : 25. **22.** 160 sq. ft. **23.** $11.25; $31.25.

Page 118. **1.** 1 : 5. **2.** 1 : 4. **3.** 2 : 3. **4.** 21 cm ; 27 cm. **5.** 40 cm ; 50 cm ; 1 : 5. **6.** 4 : 1. **7.** 4 : 3.

Page 119. **10.** 12. **11.** 20.

Page 123. **1.** 7500 sq. ft. **2.** 25 ft. **3.** 282.8 ft. ; 353.5 ft. **1.** 1 : 36. **2.** 1 : 64 ; 1 : 8.

Page 125. **1.** 16 in.; 10 in.; 2 in.; 1 in.; $\frac{1}{2}$ in.; 2 in.; $1\frac{17}{33}$ in. **2.** $3\frac{1}{4}$ mi.

Page 127. **1.** $1287\frac{1}{2}$ ft.

Page 128. **1.** 1838.5 ft.

Page 130. **1.** 142.56 ft. **2.** 124 ft.

Page 131. **1.** 75 ft. **2.** 70 ft. **3.** $205\frac{5}{7}$ ft. **4.** 40 m. **5.** 160 ft.

Page 132. **1.** 163.92 ft.

Page 133. **1.** 9.987 A.

Page 134. **1.** 15.05 A.

Page 135. **1.** 20.78 A.

Page 136. **1.** 359 A.

Page 144. **2.** 13 cm. **3.** 16 ft.

Page 148. **1.** 216 qcm. **2.** $101\frac{1}{4}$ sq. ft. **3.** 558 qcm. **4.** $56. **5.** 17,774. **6.** 622.764 sq. ft. **7.** $471\frac{3}{7}$ sq. in. **8.** $1005\frac{5}{7}$ sq. ft. **9.** $415.80. **10.** 1800 qcm. **11.** 360 qcm. **12.** 622.56 qcm. **13.** 264 sq. ft. **14.** 17.08 cm; 321.95 qcm. **15.** $113\frac{1}{7}$ sq. in. **16.** 9605.2 sq. ft. **17.** 320 sq. in. **18.** 2.64 in. **19.** 1037.7 sq. ft.

Page 149. **20.** 3401.568. **21.** 1809.5616 qcm. **22.** 154 sq. in. **23.** 154 sq. ft. **24.** 3300 sq. ft. **25.** 189.06 sq. ft. **26.** 693. **27.** $1140.48. **28.** $80,457,142\frac{6}{7}$. **29.** 4 : 1. **30.** Multiplied by 9. **31.** 2 : 1. **32.** 2 : 1. **33.** Multiplied by 4. **34.** 2 : 1. **35.** 1 : 2. **36.** 16 : 1. **37.** 27.11 in.

Page 151. **1.** 512 cu. in. **2.** 4 ft. **3.** $166\frac{3}{8}$ cu. ft.; 10,381.8 lb. **4.** 64. **5.** 2995.2 lb. **6.** 15,360. **7.** 20 in. **8.** 7202.52 lb. **9.** 12.124 cu. ft. **10.** 18 cu. ft. 72 cu. in. **11.** $5\frac{5}{21}$ cu. yd.; 8825 lb. **12.** $10,476.19. **14.** $301,186\frac{2}{7}$ cu. in.; $401,581\frac{5}{7}$ cu. in. **15.** 25,600 cu. ft.

Page 152. **16.** 935.28 cu. ft. **17.** $5749\frac{1}{3}$ cu. ft. **18.** $402\frac{2}{7}$ cu. in. **19.** 416 cu. in. **20.** 10,949.12 cu. ft. **21.** 119.16 cu. ft. **22.** $12\frac{37}{84}$. **23.** $11,498\frac{2}{3}$ cu. ft. **24.** $179\frac{2}{3}$ cu. ft. **25.** $226\frac{2}{7}$ lb. **26.** 100.1. **27.** $84,857\frac{1}{7}$ cu. in. **28.** 3 : 1. **29.** $785\frac{5}{7}$ cu. in.; $261\frac{19}{21}$ cu. in.; $523\frac{17}{21}$ cu. in.; **30.** 2156 cu. in.; $1437\frac{1}{3}$ cu. in. The sphere is just $\frac{2}{3}$ as large as the cylinder.

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The numbers refer to the pages